بِسمِ اللَّهِ الرَّحمَنِ الرَّحِيمِ

وَ عَلَى اللَّهِ فَليَتَوَكَّلِ المُتَوَكِّلُون

صَدَقَ اللَّهُ العَظِيم

Conjecture (H)

A conjectured lower bound for the cohomological dimension of elliptic spaces.

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Abstract.

Here we prove for pure spaces the following conjecture :

The sum of the Betti numbers of a 1-connected elliptic space is greater than the total rank of its homotopy groups.

Our main tool is Sullivan's minimal model.

History.

The conjecture H was formulated in 1990, by the first author in his thesis with Yves Felix (Univ. Louvain, Belgium) [Hi-90], and resolved it for pure spaces.

What about this work ?
Communications

A first part of our work was exposed as shirt talks in :

- The third Maghrebian conference of Geometry, Topology and Dynamic Systems , May 28 - June 02, 2007, Marrakech, Morocco. (http ://www.fstg-marrakech.ac.ma/colloque/ggtm.html)
- The 2007 conference of the GDR " Algebraic Topology and Applications" October 29-31, 2007 , Angers, France. (http ://math.univ-lille1.fr/ gdrtop05/Colloque2007/homepage.html).

Another part will be probably given as a communication in the

Arolla conference, Switzerland, 18-24 August 2008.
http ://sma.epfl.ch/ hessbell/arolla/.

What about this work ?
Communications missed

I was invited to give shirt talks in the following events but could not because of missing financial support.

- The Graduate Student Homotopy Conference of Cambridge, 4-7 December 2007 (http ://www.srcf.ucam.org/cugms/homotopy2007).
- The third workshop Discrete Groups and Geometric Structures, with Applications, May 26 - 30, 2008, Kortrijk, Belgium, (http ://www.kuleuven-kortrijk.be/workshop/).
- The Conference on Algebraic and Geometric Topology, 9-13 June, 2008, Gdansk, Poland (http ://math.univ.gda.pl/cagt/)

What about this work ?

Papers

- A first paper was submited to

  "Topology and its Applications"

- A second paper is now accepted in

  "Journal of Homotopy and Related Structures"

What about this work ?
Preprints

Two preprints are online in arXiv

- Title : *A conjectured lower bound for the cohomological dimension of elliptic spaces. Some results in some simple cases.*
  Authors : M.R. Hilali and M.I. Mamouni.
  ref : http ://arxiv.org/abs/0712.3784.
- Title : *The conjecture H : A lower bound of cohomologic dimension for an elliptic space*.
  Authors : M. R. Hilali and M. I. Mamouni.
  ref : http ://arxiv.org/abs/0712.3784.

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Graduation.

Graded modules.

A *graded vector space* is the data of sequence $(V^p)_{p\in\mathbb{N}}$ of $\mathbb{Q}$-vector spaces, where elements of V^p are called of degree p, i.e.,

$$x \in V^p \Longleftrightarrow |x| = p$$

We will write

$$V := \bigoplus_{p\geq 0} V^p$$

Example :

$$\mathbb{Q}[X] = \bigoplus_{p\geq 0} \mathbb{Q}X^p$$

Graduation.
cgda.

A differential algebra (A, d) is called a commutative differential graded algebra (cdga) if it verifies the following properties :

- A is graded as a module.
- $a.b = (-1)^{|a||b|} b.a$ for all $a, b \in A$.
- A is equipped with a differential d of degree 1, i.e.,

$$d : A^p \longrightarrow A^{p+1}$$

- $d(a.b) = (da).b + (-1)^{|a|} a.db$ (Leibniz rule).

In this case $y^2 = 0$ when $|y|$ is odd, and $xy = yx$ when $|x|$ is even.

Graduation.

Construction of cgda.

From any graded and differential vector space (V, d) we can construct a free cdga denoted $(\Lambda V, d)$ as follows :

- For any $p \geq 1$, we put $V^{\otimes p} := V \otimes \cdots \otimes V$ and $V^{\otimes p} := \mathbb{Q}$ when $p = 0$.
- $TV := \bigoplus_{p \geq 0} V^{\otimes p}$.
- $\Lambda V := TV \diagup (a.b - (-1)^{|a||b|} b.a)$.
- The differential on ΛV is determined by its restriction to V, respecting the Leibniz rule.

CDGA.

Properties and denotations.

- If $|x|$ is even, then $\Lambda x = \mathbb{Q}[x]$.
- If $|y|$ is odd, then $\Lambda y = \mathbb{Q}_1[y] = \mathbb{Q} \oplus \mathbb{Q}y$.
- $\Lambda(V \oplus W) = \Lambda V \otimes \Lambda W$.
- Let $V^{odd} := \bigoplus_{p\geq 0} V^{2p+1}$ and $V^{even} := \bigoplus_{p\geq 0} V^{2p}$.
  We easily check that $\Lambda V = Sym(V^{even}) \otimes Ext(V^{odd})$ is the tensor product of the symmetric algebra on V^{even} and of the exterior algebra of V^{odd}.
- $\Lambda^n V$ denotes the set of elements of ΛV of wordlength n.
- $\Lambda^{\geq n} V := \bigoplus_{k\geq n} \Lambda^k V$ denotes the set of element of ΛV of wordlength at least n.

Models of Sullivan.

A cdga $(\Lambda V, d)$ is called *Sullivan model* if V has a basis $(v_\alpha)_{\alpha \in J}$, where J is a well-ordered set, such that $dv_\beta \in \Lambda V_{<\beta}$, where $\Lambda V_{<\beta}$ is the span of $\{v_\alpha$ such that $\alpha < \beta\}$

Given a Sullivan model $(\Lambda V, d)$, we say that it is :

- minimal, when $\alpha < \beta \Longrightarrow \deg v_\alpha \leq \deg v_\beta$.
- simply connected or 1-connected, when $V^0 = V^1 = 0$, then it is minimal if and only if $dV \subset \Lambda^{\geq 2} V$.
- elliptic, when V and $H^*(\Lambda V, d)$ are both finite dimensional.
- pure, when $dV^{\text{even}} = 0$ and $dV^{\text{odd}} \subset \Lambda^{\geq 2} V^{\text{even}}$.

The basic reference of Sullivan models is [FHT01]

Euler-Poincaré characteristic.

Definitions.

For any elliptic model, we define the two followings invariants :

- Cohomological invariant : $\chi_c := \sum_{k\geq 0}(-1)^k \dim H^k(\Lambda V, d)$.
- Homotopical invariant : $\chi_\pi := \sum_{k\geq 0}(-1)^k \dim V^k$.

Euler-Poincaré characteristic.

Theorem 1. [FHT01]-page 444

Theorem 1.

If $(\Lambda V, d)$ is a simply connected and elliptic model, then

$$\chi_c \geq 0 \text{ and } \chi_\pi \leq 0$$

i.e.,

$$\dim H^{\text{even}}(\Lambda V, d) \geq \dim H^{\text{odd}}(\Lambda V, d)$$

and

$$\dim V^{\text{even}} \leq \dim V^{\text{odd}}$$

Euler-Poincaré characteristic.

Theorem 2. [FHT01]-page 444

Theorem 2.

For any simply connected and elliptic model $(\Lambda V, d)$, the following conditions are equivalent :

1. $\chi_c > 0$
2. $\chi_\pi = 0$, i.e., $\dim V^{\text{even}} = \dim V^{\text{odd}}$
3. $H^*(\Lambda V, d) = H^{\text{even}}(\Lambda V, d)$, i.e., $H^{\text{odd}}(\Lambda V, d) = 0$
4. $(\Lambda V, d)$ is isomorphic to a pure model

The Conjecture H
Elliptic spaces

Let X be a 1-connected finite CW complex, we know that $\pi_i(X)$ is a direct sum of finitely many copies of $\mathbb{Z}$ and a finite abelian group, i.e.,

$\pi_i(X) = \mathbb{Z}^{n_i} \oplus T_i$ where T_i is a finite abelian group called Torsion of $\pi_i(X)$, $n_i = \dim \pi_i(X) \otimes \mathbb{Q}$

Thus we distinguich two fundamental classes of 1-connected finite CW complexes : *elliptic* and *hyperbolic*, where elliptic spaces are those for which $\pi_i(X)$ is finite for almost all i. That will be the class which interests us in this talk. Elliptic spaces abound in homotopy theory, for example, homogeneous spaces are elliptic.

The Conjecture H
Motivation

For elliptic spaces, both

$\pi_*(X) \otimes \mathbb{Q}$ and $H^*(X; Q)$ are finite dimensional

and a

pure space is elliptic if and only if $\dim H^*(X, \mathbb{Q}) < \infty$,

so it seems roughly speaking that

$\dim H^*(X, \mathbb{Q})$ becomes larger when $\dim \pi_*(X) \otimes \mathbb{Q}$ does

That was the key idea for the first author to conjecture a lower bound for the cohomolgical dimension of an elliptic space in terms of the total rank of its homotopy groups and prove it for pur spaces [Hi-90] in 1990.

The Conjecture H
Topological version

The Conjecture H : Topological version

If X is a 1-connected elliptic space then

$$\dim H^*(X, \mathbb{Q}) \geq \dim (\pi_*(X) \otimes \mathbb{Q})$$

Let us recall that a space X is called 1-connected or simply connected when $\dim \pi_i(X) \otimes \mathbb{Q} = 0$ for $i = 0, 1$.

The Conjecture H

Sullivan models for spaces.

For any 1-connected space X of finite type, i.e., $\dim H^k(X, \mathbb{Q}) < \infty$ for all $k \geq 0$, Denis Sullivan (1970) associated a commutative differential graded algebra $(\Lambda V, d)$ (unique up to isomorphism) , called *minimal Sullivan model* of X, which algebraically models the rational homoptopy type of the space, more precisely

$$H^*(\Lambda V, d) \cong H^*(X, \mathbb{Q}) \quad \text{as algebras}$$
$$V \cong \pi_*(X) \otimes \mathbb{Q} \quad \text{as vector spaces}$$

Thus a space X is called elliptic or pure when its model does. For more details about minimal Sullivan models of spaces we refer the reader to ([FHT01], pages 138-160).

The Conjecture H
Algebraic version.

Because of the contravariant correspondance evoked here above between spaces and their minimal models, the topological version of our conjecture admits the following algebraic interpretation :

The Conjecture H : Algebraic version

If $(\Lambda V, d)$ is a 1-connected elliptic Sullivan minimal model, then

$$\dim H^*(\Lambda V, d) \geq \dim V$$

H-spaces [Ha02]

Definition

An *H-espace* is a topological space, X, provided with a continuous map, $\mu : X \times X \longrightarrow X$ such that $\mu \circ i_\varepsilon = id_X$ where $i_\varepsilon : X \longrightarrow X \times X, \varepsilon = 1, 2$ designed canonical injections.

H-spaces [Ha02]
argument

The argument was simple : The model of an H-space is of the form

$$(\land V, 0)$$

so

$$H^*(\land V, 0) \cong \land V$$

H-spaces [Ha02]

Examples I

H-spaces abound in geometry and topology, as example we have :

- Topological groups, in particular Lie groups, Homoegeneous spaces.
- The spheres : $\mathbb{S}^0, \mathbb{S}^1$, (of the complexes), $\mathbb{S}^3$, (of the quaternions), $\mathbb{S}^7$, (of the octanions).
  Adams showed that there are the only spheres which are H-spaces.
- The projectif spaces : $\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^7$.
  In the general case $\mathbb{R}\mathrm{P}^n$ is an H-space if and only if $n+1$ is a power of 2.

H-spaces [Ha02]

Examples II

- The loop spaces : ΩX, for a based space X.
- The Eilenberg-MacLane spaces : $K(G, n)$, with $n \geq 1$ and G abelien.

  Definition : X is a $K(G, n)$ if it is solution of the problem :

  $$\pi_i(X) = \begin{array}{l} G \text{ if } i = n \\ 0 \text{ if } i \neq n \end{array}$$

  Argument : $K(G, n) = \Omega K(G, n+1)$.

H-spaces [Ha02]
Examples III

- The James reduced product associated to a topological based space, $(X, *)$, defined by the relation :

$$J(X) := \coprod_{k \geq 1} X^k \diagup (x_i = *)$$

- The infinite symmetric product associated to a topological space, $(X, *)$, defined by the relation :

$$SP(X) := \coprod_{k \geq 1} X^k \diagup (x_1, \ldots, x_k) \sim (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

Hyperelliptic spaces

An elliptic 1-connected space X and its minimal model $(\Lambda V, d)$ are called hyperelliptic if

$$dV^{\text{even}} = 0$$
$$dV^{\text{odd}} \subset \wedge^+ V^{\text{even}} \otimes \wedge V^{\text{odd}}$$

We have shown that the conjecture H, holds for hyperelliptic spaces under the condition

$$\chi_c \geq \frac{1 + \sqrt{-12\chi_\pi - 15}}{2}$$

Toral Rank

The action of group G on a space is called almost free when all isotropy groups $G_x = \{g \in G \text{ such that } g.x = x\}$ are finite.

$rk_0(X)$: The toral rank of a space X is the maximum of n such that the toric $\mathbb{T}^n$ acts almost freely on X.

The conjecture H holds when $rk_0(X) \in \{-\chi_\pi - i\},\ i = 0, 1, 2$.

Formal Dimension

For an elliptic space, X, the largest integer k such that $H^k(X,\mathbb{Q}) \neq 0$, is called formal dimension of X and denoted $fd(X)$

The conjecture H holds when

$$(fd(X) \leq 10) \text{ or } (fd(X) - rk_0(X) \leq 6).$$

Symplectic manifolds

The conjecture H holds for symplectic manifolds.

A symplectic manifold is a smooth manifold M equipped with a closed, nondegenerate, 2-form ω called the symplectic form.

Kahler $\Longrightarrow$ symplectic $\Longrightarrow$ Poisson $\Longrightarrow$ Jacobi' manifolds.

coymplectic manifolds

The conjecture H holds for cosymplectic manifolds.

A cosymplectic manifold structure on a $2k+1$-dimension manifold is the data of a closed 1-form θ and a closed 2-form ω such as $\theta \wedge \omega^k$ is a volume form on M, where ω^k indicates the product of k copies of ω.

Differential of homegeneous-length

We say that ΛV has differential d of homegeneous-length k if

$$dV \subset \Lambda^k V$$

We have shown that the conjecture H holds in the following cases :

- If $(\Lambda V, d)$ is an elliptic minimal model with an homogeneous-length differential and whose rational Hurewicz homorphism is non-zero in some odd degree.
- If $(\Lambda V, d)$ is an elliptic minimal model with an homogeneous-length differential at least 3.
- If $(\Lambda V, d)$ is an elliptic minimal model with a differential, homogeneous of length 2 (i.e : coformal) with odd degree generators only, (i.e., $V^{\text{even}} = 0$)

Pure spaces I
Proof

We know from ([FHT01]-Proposition 32.10- page 444) that $H^*(\wedge V, d) = H^{\text{even}}(\wedge V, d)$ and that $\dim V^{\text{even}} = \dim V^{\text{odd}}$. Then consider $\{x_1, \cdots, x_n\}$ an homgeneous basis for V^{even} and $\{y_1, \cdots, y_n\}$ an homgeneous basis for V^{odd}. Denote by W_1 and W_2 the vector subspaces of $H^{\text{even}}(\wedge V, d)$ generated respectivelly by $([x_i])_{1\leq i\leq n}$ and $([x_i x_j])_{1\leq i\leq j\leq n}$. Put $W_0 = H^0(\wedge V, d) \cong \mathbb{Q}$. Minimality of the model ensure that $W_0 \oplus W_1 \oplus W_2$ is a direct sum in $H^{\text{even}}(\wedge V, d)$ and that $([x_i])_{1\leq i\leq n}$ are linearly independent, so

$$\dim H^{\text{even}}(\wedge V, d) \geq 1 + n + \dim W_2$$

Pure spaces II
Proof

On the other hand

$$W_2 \oplus (\Lambda^2 V^{\text{even}} \cap dV^{\text{odd}}) = \Lambda^2 V^{\text{even}}$$
$$\dim \Lambda^2 V^{\text{even}} = \frac{n(n+1)}{2}$$
$$\dim(\Lambda^2 V^{\text{even}} \cap dV^{\text{odd}}) \leq n$$
$$\dim W_2 \geq \frac{n(n+1)}{2} - n$$

Thus

$$\begin{aligned} \dim H^*(\Lambda V, d) = & \dim H^{\text{even}}(\Lambda V, d) \\ \geq & \frac{n(n+1)}{2} + 1 \\ \geq & 2n = \dim V \end{aligned}$$

□

Formal spaces

We have shown that the conjecture H holds for formal spaces.

A commutative differential graded algebra A is called formal if $A^0 = \mathbb{Q}$ and it has the same minimal Sullivan model as a commutative differential graded algebra with vanishing differential. A model $(\Lambda V, d)$ is called formal when its cohomology $H^*(\Lambda V, d)$ is formal. A space is called formal if its minimal Sullivan model is formal.
Examples of formal spaces include spheres, H-spaces, symmetric spaces, and compact Kähler manifolds. Formality is preserved under wedge sums and direct products ; it is also preserved under connected sums for manifolds.
Steven Halperin and Jim Stasheff has given an algorithm for deciding whether or not a space is formal.

Main References

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