RATIONAL HOMOTOPY THEORY SEMINAR

Sullivan's

Minimal Models

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Aim of the talk

- $\rightsquigarrow$ **Rational Homotopy Theory** : Brief description
- $\rightsquigarrow$ **Modèle minimal de Sullivan** : Definition and main proprieties

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Prerequisites

Algebraic Topology basic knowledge of what is :

- ⇝ **Homology**
- ⇝ **Homotopy**

Prerequisites

Graded & differential

(A, d) that means :

- $\rightsquigarrow$ A, module, vector space, algebra, ...
- $\rightsquigarrow$ $A := \bigoplus_k A^k$
- $\rightsquigarrow$ $d_k : A^k \longrightarrow A^{k+1}$ such that $d_{k+1} \circ d_k = 0$
- $\rightsquigarrow$ We write $d : A \longrightarrow A$, where $d_k = d|_{A_k}$ and $d^2 = 0$

Prerequisites

Cohomology

(A, d) that means :

- $\rightsquigarrow$ $H^k(A, d) := \dfrac{\ker d_{k+1}}{\mathrm{Im} d_k}$
- $\rightsquigarrow$ $H^*(A, d) := \bigoplus_k H^k(A, d)$

♣

Prerequisites

Homotopy

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- $\rightsquigarrow$ Two continuous maps $f, g : \mathbb{S}^k \longrightarrow X$ are called homotopic when there is continuous deformation $H : \mathbb{S}^k \times [0,1] \longrightarrow X$, such that $H(.,0) = f, H(.,1) = g$. We obtain an equivalence relation $\sim$
- $\rightsquigarrow$ The homotopy groups : $\pi_k(X) := \mathcal{C}(\mathbb{S}^n, X)/\sim$ and $\pi_*(X) := \bigoplus_k \pi_k(X)$.
- $\rightsquigarrow$ Type of homotopy : X and Y are called with the same type of homotopy when $\pi_*(X) \cong \pi_*(Y)$.

♣

Glossary

Rational Space

A simply connected space X is called rational if the following is satisfied.

$$\pi_*(X) \text{ is a } \mathbb{Q}\text{-vector space.}$$

N.B : $\pi_*(X) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-vector space

Glossary

Rationalization

Let X be a simply connected space. A rationalization of X is simply connected and rational space Y, such that :

$$\pi_*(X) \otimes Q \cong \pi_*(Y)$$
$$H^*(X;\mathbb{Q}) \cong H^*(Y;\mathbb{Q})$$

Glossary

Theorem, [FHT]

Any simply connected space X admits an unique (up to homotopy) CW-complex rationalization

Glossary

Definition

The rational homotopy type of a simply connected space X is the homotopy type of its rationalization.

What it is it

Rational homotopy theory is the study of rational homotopy types of spaces and of the properties of spaces and maps that are invariant under rational homotopy equivalence.

Founders in 1967

Denis Sullivan (1941-), CUNY-SUNY, USA

work in topology, both algebraic and geometric, and on dynamical systems
Doctoral advisor : William Browder
Wolf Prize in Mathematics (2010)
Leroy P. Steele Prize (2006)
National Medal of Science (2004)

Founders in 1967

Daniel Quillen (1940-2011), Oxford

the "prime architect" of higher algebraic K-theory
Doctoral advisor : Raoul Bott
Fields Medal (1978)
Cole Prize (1975)
Putnam Fellow (1959)

Model of Sullivan

CGDA

A commutative graded differential algebra over the rational numbers is a graded $\mathbb{Q}$-algebra (A, d) such that

$$\begin{cases} & ab = (-1)^{|a||b|}ba \\ (*) & d(ab) = (da).b + (-1)^{|a|}b.da \quad \text{for all } a, b \in A \end{cases}$$

In particular :

- $\rightsquigarrow$ $y^2 = 0$ when $|y|$ odd
- $\rightsquigarrow$ $xy = yx$ when $|x|$ even

Model of Sullivan

How to build it

From any differential and graded $\mathbb{Q}$-vector space V, we define the cgda ΛV denotes defined by

$$\Lambda V = TV \diagup \langle v \otimes w - (-1)^{|v||w|} w \otimes v \rangle$$

where TV denotes the tensor algebra over V.
The differential on ΛV is naturally extended from that of V with respecting the condition (*) called of nilpotence or of Leibniz

♣

Model of Sullivan

Our cgda is called a model of Sullivan when there exists some well ordered basis $(v_\alpha)_{\alpha\in I}$ of V such that

$$dv_\alpha \in \Lambda\{v_\beta,\ \beta < \alpha\}$$

Model of Sullivan

Minimal model

The model of Sullivan is called minimal when

$$\alpha < \beta \Longrightarrow |v_\alpha| \leq |v_\beta|$$

♣

Model of Sullivan

Elliptic model

The minimal model is called elliptic when both V and $H^*(\Lambda V, d)$ are finite dimensional, in this case

$$(\Lambda V, d) = (\Lambda\{x_1, \ldots, x_n\}, d) \quad \text{with} \quad |x_1| \leq \ldots |x_n|$$

$$dx_1 = 0$$

$$\text{and} \quad dx_j \in \Lambda(x_1, \ldots, x_{j-1}) \text{ for } j \geq 2$$

♣

Model of Sullivan

D. Sullivan, [Su]

Any simply connected space have a minimal model of Sullivan, $(\Lambda V, d)$ (unique up to isomorphism of cgda), who models its cohomology and homotopy as follows :

$$\begin{array}{rcl} H^k(X;\mathbb{Q}) & \cong & H^k(\Lambda V, d) \\ \pi_k(X)\otimes\mathbb{Q} & \cong & V^k \end{array}$$

♣

Model of Sullivan

Basic Examples

For the odd sphere : $\mathbb{S}^{2k+1}$, the model is the form $(\Lambda\{x\}, 0)$ with $|x| = 2k+1$. So

$$\begin{aligned} \pi_n(\mathbb{S}^{2k+1}) &\cong \mathbb{Z} \quad \text{if } n = 2k+1 \\ &\cong 0 \quad \text{if not} \end{aligned}$$

Model of Sullivan

Basic Examples

For the even sphere : $\mathbb{S}^{2k}$, the model is the form $(\Lambda\{x, y\}, d)$ with $|x| = 2k, |y| = 4k - 1, dy = x^2$. So

$$\begin{aligned} \pi_n(\mathbb{S}^{2k}) &\cong \mathbb{Z} \quad \text{if } n = 2k \\ &\cong \mathbb{Z} \quad \text{if } n = 4k - 1 \\ &\cong 0 \quad \text{if not} \end{aligned}$$

♣

Special Denotations

- In general for any $x \in \Lambda V$, we have

$$dx = \sum_{k \geq 0} \beta_k \overbrace{\underbrace{y_1 \dots y_k}_{|y_i| \text{ odd}}}^{\text{lenght}=k} \prod_{|x_i| \text{ even}} x_i^{\alpha_i} \in \bigoplus_k \Lambda^k V^{\text{odd}} \otimes \Lambda V^{\text{even}}$$

- Hence ΛV is bi-graded as follows $\Lambda V = \bigoplus_{p,q} (\Lambda^p V^{\text{odd}} \otimes \Lambda V^{\text{even}})^q$. p word-length graduation and q : degree graduation.
- $\Lambda^{\geq k} V := \bigoplus_{p \geq k} \Lambda^p V$ and $\Lambda^+ V := \Lambda^{\geq 1} V$
- When $(\Lambda V, d)$ is a simply elliptic minimal model, we have

$$dV \subset \Lambda^{\geq 2} V = \Lambda^+ V . \Lambda^+ V$$

Special Denotations

Simple Conclusions

- $\rightsquigarrow$ $\Lambda(V \oplus W) = \Lambda V \otimes \Lambda W$
- $\rightsquigarrow$ When $|y|$ odd, $\Lambda y = \{ay + b; a, b \in \mathbb{Q}\} = \mathbb{Q}_1[y]$
- $\rightsquigarrow$ When $|x|$ even, $\Lambda x = \{\sum_k a_k x^k; a, b \in \mathbb{Q}\} = \mathbb{Q}[x]$

♣

Special Denotations

Pure Model

When

$$\begin{aligned} dV^{\text{even}} &= 0 \\ dV^{\text{odd}} &\subset \Lambda V^{\text{even}} \end{aligned}$$

♣

Special Denotations

Hyperelliptic Model

When

$$dV^{\text{even}} = 0$$

Special Denotations

Two Stage Model

When

$$\begin{aligned} V &= U \oplus W \\ dU &= 0 \\ dW &\subset \Lambda U \end{aligned}$$

Example of Algebraization

Hilali Conjecture (1990)

For any elliptic and simply connected topological space X, we have

$$\dim(\pi_*(X) \otimes \mathbb{Q}) \leq \dim H^*(X; \mathbb{Q})$$

Example of Algebraization

Algebraic version

For any elliptic model of Sullivan, $(\Lambda V, d)$ we have

$$\dim V \leq \dim H^*(\Lambda V, d)$$

♣

Example of Algebraization

Simple example in which it holds

For the sphere $\mathbb{S}^n$ we have seen that

$$\dim V = 1 \text{ or } 2$$

, and its well known that

$$H^0(\mathbb{S}^n;\mathbb{Q}) = H^n(\mathbb{S}^n;\mathbb{Q}) = \mathbb{Q} \text{ and } H_i(X;\mathbb{Q}) = 0$$

for all other i.

Euler-Poincaré characteristic

Definition

For any 1-connected elliptic model $(\Lambda V, d)$ we define two invariants. One cohomological :

$$\chi_c := \sum_{k\geq 0}(-1)^k \dim H^k(\Lambda V, d)$$

and another homotopic :

$$\chi_\pi := \sum_{k\geq 0}(-1)^k \dim(V^k)$$

♣

Euler-Poincaré characteristic

S. Halperin, [Ha83]

we have the following :

$$\chi_c \geq 0 \text{ and } \chi_\pi \leq 0$$

Morever,

$$\chi_c > 0 \Longleftrightarrow \chi_\pi = 0$$

In this case

$$H^*(\Lambda V, d) = H^{\text{even}}(\Lambda V, d)$$

♣

Euler-Poincaré characteristic

Generalisation

For any graded vector space A, the Euler-Poincaré characteristic is defined as follows

$$\chi(A) := \sum_{k \geq 0} (-1)^k \dim A^k$$

So,

$$\chi_c = \chi(H^*(\Lambda V, d)) , \ \chi_\pi = \chi(V)$$

N.B

$$\chi(H^*(A, d)) = \chi(A)$$

Euler-Poincaré characteristic

Util Remark

- As $\chi_\pi = \dim V^{\text{even}} - \dim V^{\text{odd}}$, we put $\dim V^{\text{even}} = p$ and $\dim V^{\text{odd}} = n + p$, so
- $\chi_\pi = -p$ and $\dim V = 2n + p$
- $p = 0 \Longleftrightarrow H^*(\Lambda V, d) = H^{\text{even}}(\Lambda V, d)$
- $p \neq 0 \Longleftrightarrow \dim H^*(\Lambda V, d) = 2 \dim H^{\text{even}}(\Lambda V, d)$

Toral Rank

Definition

$rk_0(X) :=$ The largest integer $n \geq 1$ for which X admits an almost-free n-torus action

C. Allday & Halperin, [AH78]

$$rk_0(X) \leq -\chi_\pi$$

The equality holds when X is pure

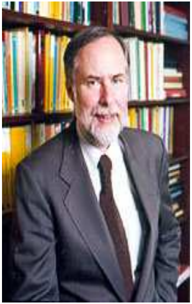
Toral Rank

Toral Rank

Toral Rank Conjecture (TRC), S.Halperin (1986)

For any elliptic and simply connected topological space X, we have

$$\dim H^*(X;\mathbb{Q}) \geq 2^{rk_0(X)}$$



Toral Rank

The link between ConjH & TRC

- We know by [AH78] that $rk_0(X) \leq -\chi_\pi = p$, the write

$$\begin{array}{ll} \text{TRC :} & \dim H^*(X;\mathbb{Q}) \geq 2^{p-\varepsilon} \\ \text{Conj. H :} & \dim H^*(X;\mathbb{Q}) \geq 2n + p \end{array}$$

♣

- Conj H $+\ 2n + p \geq 2^{p-\varepsilon} \Longrightarrow$ CRT

Formal dimension

Definition

For an elliptic space X, we put

$$fd(X) := \max\{k,\ H^k(X,\mathbb{Q}) \neq 0\}$$

Formal dimension

J. Friedlander and S. Halperin, [FH79]

If X is a 1-connected and elliptic space of minimal Sullivan model $(\Lambda V, d)$, then $fd(X) \geq \dim V$

Formal dimension

Best known result, losed source

If X is a 1-connected and elliptic manifold, then $fd(X) = \dim X$ ♣

Formal dimension

J. Friedlander and S. Halperin, [FH79]

There exists a special homogeneous basis $x_1, \ldots, x_n$ of V^{even} and a basis $y_1, \ldots y_{n+p}$ of V^{odd} such that :

- $\rightsquigarrow$ $|x_1| \leq \cdots \leq |x_n|$
- $\rightsquigarrow$ $2|x_i| - 1 \leq |y_i|$ for $i = 1..n$
- $\rightsquigarrow$ $\displaystyle\sum_{i=1}^{n} |x_i| \leq fd(X)$
- $\rightsquigarrow$ $\displaystyle\sum_{i=1}^{n+p} |y_i| \leq 2fd(X) - 1$
- $\rightsquigarrow$ $\displaystyle\sum_{i=1}^{n+p} |y_i| - \sum_{i}^{n}(|x_i| - 1) = fd(X)$

Main References of RHT

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THANK YOU!
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