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## Ganea Conjecture for TC

### Réferences :

- Rational Topological complexity , B. Jessup, A. Muñoz, P.E. Parent , « Algebraic & Geometric topology » (20xx) 1001-999».
- A proof of Ganea conjecture for rational spaces, Kathryn P. Hess, Topology Vol. 30 No 2 pp 205-214 1991.
- Rational LS Category and a conjecture of Ganea, Barry Jessup, Journal of Pure and Applied Algebra 65 (1990) 57-67.

# Introduction :

The Ganea Conjecture for LS-Category stated that the LS-Category for a space increases by one when taking a product with a sphere.

This conjecture was proved to be false in the general case by N. Iwase, but it remains true in the rational category by Kathryn P. Hess.

We will introduce here ~~an application of~~ the Ganea Conjecture for rational topological complexity and we will prove it by using mtc. So the conjecture is:

$$\{ \text{TCC}(X \times S^n) = \text{TC}(X) + \text{TC}(S^n) \}$$

Let  $(\Lambda V, d)$  be a Sullivan model of  $X$  and let  $K \subset \Lambda V \otimes \Lambda V$  be the Kernel of the multiplication  $\mu : \Lambda V \otimes \Lambda V \longrightarrow \Lambda V$ . For any  $n \geq 1$ , denote by  $K^n$  the  $n^{\text{th}}$  power of  $K$  i.e. the ideal generated by products of elements of  $K$  of length at least  $n$ .

**Definition 8** Consider the projection

$$\Lambda V \otimes \Lambda V \xrightarrow{p_m} \Lambda V \otimes \Lambda V / K^{m+1}$$

Then :

- (i)  $tc(X)$  is the smallest  $m$  for which  $p_m$  has a homotopy retraction as algebras.
- (ii)  $m tc(X)$  is the smallest  $m$  for which  $p_m$  has a homotopy retraction as  $(\Lambda V \otimes \Lambda V)$ -modules.

**Recalling :**

Having a homotopy retraction means that in any Sullivan model of the projection

$$\begin{array}{ccc} \Lambda V \otimes \Lambda V & \xrightarrow{i} & \Lambda V \otimes \Lambda V \otimes \Lambda W \\ & \dashleftarrow p & \downarrow \cong \\ & & \Lambda V \otimes \Lambda V / K^{m+1} \end{array}$$

The map  $i$  has a retraction  $p : \Lambda V \otimes \Lambda V \otimes \Lambda W \longrightarrow \Lambda V \otimes \Lambda V$  which is a map of differential algebras or  $(\Lambda V \otimes \Lambda V)$ -modules.

Let  $(\Lambda V, d)$  be a Sullivan model of  $X$ . The multiplication map of  $\Lambda V$ ,  $\mu : (\Lambda V, d) \otimes (\Lambda V, d) \longrightarrow (\Lambda V, d)$  is a model of the diagonal  $\Delta : X \longrightarrow X \times X$ . Let  $A$  denote  $(\Lambda V, d) \otimes (\Lambda V, d)$  and  $K \subset A$  the kernel of  $\mu$ . Then, a model of the  $m^{\text{th}}$  fat wedge associated to the diagonal  $\Delta : X \longrightarrow X \times X$  is given by the projection

$$A^{\otimes m+1} \xrightarrow{p} A^{\otimes m+1}/K^{\otimes m+1}$$

If  $A^{\otimes m+1} \xrightarrow{M} A$  denotes the iterated multiplication  $x_1 \otimes \dots \otimes x_{m+1} \mapsto x_1 \dots x_{m+1}$ , and  $A^{\otimes m+1} \xrightarrow{i} A^{\otimes m+1} \otimes \Lambda W \xrightarrow{\cong} A^{\otimes m+1}/K^{\otimes m+1}$  is a relative Sullivan model of  $p$ , then we have the following proposition:

### Proposition

(a)  $\text{TC}(X_{\otimes})$  is the least  $m$  for which there is a map of differential graded algebras

$$\rho : A^{\otimes m+1} \otimes \Lambda W \longrightarrow A \quad \text{with } \rho j = M$$

(b)  $\text{MTC}(X)$  is the least  $m$  for which there is a map of differential graded  $A$ -modules  $\rho : A^{\otimes m+1} \otimes \Lambda W \rightarrow A$  with  $\rho j = M$

Hence forth, we use (a) and (b) above as definitions of  $\text{TC}$  and  $\text{MTC}$ .

**Proposition** For any simply connected space  $X$ ,

$$MTC(X) \leq TC(X_{\#}) \leq tc(X)$$

and

$$MTC(X) \leq mtc(X) \leq tc(X)$$

**Proof**

with the notation of the above, simply note that the multiplication map  $M$  above takes  $K^{\otimes m+1}$  to  $K^{m+1}$ , and so induces a map of differential graded algebras

$$A^{\otimes m+1}/K^{\otimes m+1} \xrightarrow{\tilde{M}} A/K^{m+1}$$

Thus, any homotopy retraction of  $A \rightarrow A/K^{m+1}$  will, essentially by precomposition with  $\tilde{M}$ , induce the desired map  $\rho$ .

**Lemma** Suppose  $X$  and  $Y$  are well-pointed.

If  $X \vee Y \rightarrow X \times Y$  is a homotopy equivalence, then

$$TC(X \vee Y) \leq \max\{TC(X), TC(Y)\}$$

**Proof** If we regard  $X \vee Y$  as the subset  $X \times \{y_0\} \cup \{x_0\} \times Y$  of the product  $X \times Y$ , then  $(X \vee Y) \times (X \vee Y)$  is the union of the following subset of  $X \times Y \times X \times Y$ :

$$(X \vee Y) \times (X \vee Y) = X \times \{y_0\} \times X \times \{y_0\} \cup X \times \{y_0\} \times \{x_0\} \times Y \cup \\ \cup \{x_0\} \times Y \times X \times \{y_0\} \cup \{x_0\} \times Y \times \{x_0\} \times Y.$$

On the other hand, since our spaces are well pointed, the inclusion  $X \vee Y \hookrightarrow X \times Y$  is a cofibration, and thus, as it is a homotopy equivalence, it is also a deformation retract. Hence we use the deformation retraction of  $X \times Y$  onto  $X \times \{y_0\} \cup \{x_0\} \times Y$  to deduce that

$$X \times \{y_0\} \times \{x_0\} \times \{y_0\} \cup \{x_0\} \times \{y_0\} \times \{x_0\} \times Y \quad \text{and}$$

$\{x_0\} \times \{y_0\} \times X \times \{y_0\} \cup \{x_0\} \times Y \times \{x_0\} \times \{y_0\}$  are also deformation retracts of  $X \times \{y_0\} \times \{x_0\} \times Y$  and  $\{x_0\} \times Y \times X \times \{y_0\}$  respectively. Therefore,

$$(X \times X) \vee (Y \times Y) = X \times \{y_0\} \times X \times \{y_0\} \cup \{x_0\} \times Y \times \{x_0\} \times Y$$

is a deformation retract of  $(X \vee Y) \times (X \vee Y)$ . We denote this retraction by  $r : (X \times X) \vee (Y \times Y) \xleftarrow[r]{\cong} (X \vee Y) \times (X \vee Y)$

Now, suppose we have homotopy sections of the fibrations

$X^I \longrightarrow X \times X$  and  $Y^I \longrightarrow Y \times Y$  over coverings  $\{U_1, \dots, U_n\}$  and  $\{V_1, \dots, V_m\}$  of  $X \times X$  and  $Y \times Y$  respectively.

Assume  $n \geq m$ . Using the homotopy lifting property, we may assume these sections to be base point preserving whenever any of the elements of these coverings contain the base point. Thus, there are homotopy sections of

$$(X \vee Y)^I \longrightarrow (X \vee Y) \times (X \vee Y) \xrightarrow{\sim} (X \times X) \vee (Y \times Y)$$

over the covering  $\{U_1 \vee V_1, \dots, U_m \vee V_m, U_{m+1}, \dots, U_n\}$ .

Finally, consider the induced homotopy sections of

$$(X \vee Y)^I \longrightarrow (X \vee Y) \times (X \vee Y) \text{ over the covering}$$

$$\{r^{-1}(U_1 \vee V_1), \dots, r^{-1}(U_m \vee V_m), r^{-1}(U_{m+1}), \dots, r^{-1}(U_n)\}.$$

If  $X = \mathbb{S}^2 \bigcup_f e^3$  and  $Y = \mathbb{S}^2 \bigcup_g e^3$ , where  $f$  and  $g$  are maps of degree 2 and 3 respectively, then  $TC(X)$  and  $TC(Y)$  are positive, and  $X \vee Y \hookrightarrow X \times Y$  is a homotopy equivalence.

Then, by all of the above, this is also a deformation retract  $\mathbb{H}$ , and by the previous lemma, we have:

$$TC(X \times Y) = TC(X \vee Y) \leq \max\{TC(X), TC(Y)\} < TC(X) + TC(Y)$$

As stated in the introduction, it is open whether, for a rational simply connected cw-complex  $X$  of finite type, one always has the equality  $TC(X \times \mathbb{S}^n) = TC(X) + TC(\mathbb{S}^n)$ .

However, this Ganea formula does hold for  $mTC$ , as we prove in the next theorem. Note that, for  $\mathbb{S}^n$ :

$$MTC(\mathbb{S}^n) = mTC(\mathbb{S}^n) = TC(\mathbb{S}_\Phi^n) = TC(\mathbb{S}^n) - TCC(\mathbb{S}^n) = \begin{cases} 1 & \text{if } n \text{ odd} \\ 2 & \text{if } n \text{ even} \end{cases}$$

The first three equalities trivially hold as  $\mathbb{S}^n$  is a formal space while the fourth is well known. For the second note that  $TC(\mathbb{S}_\Phi^n) = \text{nil Ker } U_\Phi = TCC(\mathbb{S}^n)$ .

**Theorem** If  $X$  is simply connected cw-complex of finite type and  $n \geq 2$ , then

$$MTC(X \times S^n) \geq MTC(X) + MTC(S^n) = mtc(X) + TC(S^n)$$

Moreover

$$\begin{aligned} mtc(X \times S^n) &= mtc(X) + mtc(S^n) = mtc(X) + MTC(S^n) \\ &= mtc(X) + TC(S^n) \end{aligned}$$

**Proof**

We first prove subadditivity of  $mtc$ , i.e.,

$$mtc(X \times Y) \leq mtc(X) + mtc(Y)$$

Let  $\Lambda V$  and  $\Lambda W$  be Sullivan models of  $X$  and  $Y$ .

Write  $V \oplus V = V^2$  and observe that, if  $K_V$  denotes the kernel of the multiplication  $\Lambda V^2 \rightarrow \Lambda V$ , then

$K_{V \oplus W}$  is generated as a  $\Lambda(V \oplus W)$ -module by

$\{v \otimes 1 - 1 \otimes v, w \otimes 1 - 1 \otimes w / v \in V, w \in W\}$  (by the previous lemma).

Thus, for  $m, n \geq 1$ , there is a natural morphism of algebras

$$\Lambda(V^2 \oplus W^2) / K_{V \oplus W}^{m+n+1} \longrightarrow \Lambda(V^2) / K_V^{m+1} \otimes \Lambda(W^2) / K_W^{n+1}$$

which induces a morphism  $h$  between the Sullivan models of the quotients:

$$\begin{array}{ccc} \Lambda(V^2 \oplus W^2) & \xrightarrow{i} & \Lambda V^2 \otimes \Lambda R \otimes \Lambda W^2 \otimes \Lambda S \\ \downarrow h & & \downarrow \cong \\ \Lambda(V^2 \oplus W^2) \otimes \Lambda T & \xrightarrow{\cong} & \Lambda(V^2) / K_V^{m+1} \otimes \Lambda(W^2) / K_W^{n+1} \end{array}$$

Thus, if  $j$  has a retraction  $\rho$  (either as a morphism of algebras or  $\Lambda(V \oplus W)$ -modules), then  $\rho h$  is a retraction of  $i$ .

This proves the assertion and also the subadditivity of "rational" TC.

We now prove the reverse inequality for mtc whenever  $Y$  is a sphere. Again, let  $(\Lambda V, d)$  be a Sullivan model of  $X$  and  $(\Lambda U_n, d)$  denote the model of an  $n$ -sphere  $S^n$ , so that  $U_{2k+1} = \text{span}\{u\}$ ,  $U_{2k} = \text{span}\{u, y\}$ ,  $du = 0 = dn$ ,  $dy = u^2$ ,  $|u|$  is odd and  $|y|$  is even. In what follows, we suppress the dependence on  $n$  wherever possible.

Let  $A = \Lambda V \otimes \Lambda V$ ,  $C = \Lambda U_n \otimes \Lambda U_n$ , and  $B = A \otimes C$ , all differentials being those from the products.

Denote by  $K$  and  $L$  the kernel of the multiplication in  $A$  and  $B$  respectively. Consider  $C$  the element  $z \otimes 1 - 1 \otimes z \in U_n \oplus U_n$  denoted by  $z - z'$  henceforth, and observe that  $z - z' \in L$ .

Now define  $\gamma = \begin{cases} u - u' & \text{if } n \text{ is odd} \\ (u - u')^2 & \text{if } n \text{ is even} \end{cases}$

Note that  $\gamma \in \begin{cases} L & \text{if } n \text{ is odd} \\ L^2 & \text{if } n \text{ is even} \end{cases}$

and that,  $[\gamma] \neq 0$  in  $H^*(C)$ , and thus it's also nonzero in  $H^*(B)$ .

Hence, as graded differential vector spaces, we may write

$$C = \langle \gamma \rangle \oplus M$$

and we can define a map  $p: C \rightarrow \mathcal{S}$  of graded differential vector spaces by  $p(k\gamma + m) = k$ .

Note that  $p$  is homogeneous of degree  $-|\gamma|$ .

This allows us to write  $B = (A \otimes \langle \gamma \rangle) \oplus (A \otimes M)$ , the direct sum being now one of differential  $A$ -modules.

Now define two maps of differential  $A$ -modules  $\sigma: A \rightarrow B$  and  $\tau: B \rightarrow A$  by  $\sigma(x) = x\gamma$  and  $\tau(x_0\gamma + x_1m) = x_0$ .

It is easy to check that, for all  $m$ ,  $\sigma(K^m) \subset L^{m+mtc(\$^n)}$ , which implies that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ \downarrow p_{m+1} & & \downarrow \tau \\ A/K^{m+1} & \xrightarrow{\bar{\sigma}} & B/L^{m+mtc(\$^n)+1} \end{array}$$

Moreover,  $\tau$  is a retraction of  $\sigma$ . Thus if  $\mathcal{S}$  has a homotopy retraction  $p_{m+1}$  does as well. This prove that

$$mtc(X) + mtc(\$^n) \leq mtc(X \times \$^n)$$

and so establishes the Ganea formula for  $mtc$ .

The inequality  $MTC(X \times \$^n) \geq MTC(X) + TCC(\$^n)$  is established in a similar way, which we now outline. We use the same notation as before. Briefly the map:

$$\bar{\sigma}: A^{\otimes m+1} \longrightarrow B^{\otimes m+MTC(\$^n)+1} \quad \text{defined by:}$$

$$\alpha_1 \otimes \dots \otimes \alpha_{m+1} \longrightarrow \begin{cases} \alpha_1 \otimes \dots \otimes \alpha_{m+1} \otimes (u-u'), & n \text{ odd} \\ \alpha_1 \otimes \dots \otimes \alpha_{m+1} \otimes (x-x') \otimes (u-u'), & n \text{ even.} \end{cases}$$

Satisfies  $\mathcal{F}(K^{\otimes m+1}) \subset L^{\otimes m+MTC(\$^n)+1}$  and so induces a commutative diagram

$$\begin{array}{ccccc}
 A^{\otimes m+1} & \xrightarrow{\mathcal{F}} & B^{\otimes m+MTC(\$^n)+1} & & \\
 \downarrow p_A & & \downarrow p_B & & \nearrow \rho \\
 A^{\otimes m+1}/K^{\otimes m+1} & \xrightarrow{\cong} & B^{\otimes m+MTC(\$^n)+2} & \not\cong & B^{\otimes m+MTC(\$^n)+1}
 \end{array}$$

Where  $M_B$  is multiplication. Thus, if there is (up to a relative Sullivan model) a map  $\rho$  as shown with  $\rho p_B \simeq M_B$ , post composition with  $\cong$  defined earlier shows that

$$MTC(X) \leq MTC(X \times \$^n) - MTC(\$^n).$$

The end.