Lecture Notes in Algebraic Topology

James F. Davis

Paul Kirk

Author address:

Department of Mathematics, Indiana University, Bloomington, IN 47405

E-mail address: jfdavis@indiana.edu, pkirk@indiana.edu

Chapter 4

Fiber Bundles

Fiber bundles form a nice class of maps in topology, and many naturally occurring maps are fiber bundles. A theorem of Hurewicz says that fiber bundles are fibrations, and fibrations are a natural class of maps to study in algebraic topology, as we will soon see. There are several alternate notions of fiber bundles and their relationships with one another is somewhat technical. The standard reference is Steenrod's book [**37**].

A fiber bundle is also called a Hurewicz fiber bundle or a locally trivial fiber bundle. The word "fiber" is often spelled "fibre," even by people who live in English speaking countries in the Western hemisphere.

4.1. Group actions

Let G be a *topological group*. This means that G is a topological space and also a group so that the multiplication map $\mu : G \times G \to G, \mu(g,h) = gh$ and the inversion map $\iota : G \to G, \iota(g) = g^{-1}$ are continuous.

Definition 4.1. A *topological group* G *acts on a space* X if there is a group homomorphism $G \to \mathrm{Homeo}(X)$ such that the "adjoint"

$$G \times X \to X \qquad (g,x) \mapsto g(x)$$

is continuous. We will usually write $g \cdot x$ instead of $g(x)$.

The *orbit* of a point $x \in X$ is the set $Gx = \{g \cdot x | g \in G\}$.

The *orbit space* or *quotient space* X/G is the quotient space $X/{\sim}$, with the equivalence relation $x {\sim} g \cdot x$.

The *fixed set* is $X^G = \{x \in X | g \cdot x = x \text{ for all } g \in G\}$.

An action is called *free* if $g(x) \neq x$ for all $x \in X$ and for all $g \neq e$.

An action is called *effective* if the homomorphism $G \to \mathrm{Homeo}(X)$ is injective.

A variant of this definition requires the homomorphism $G \to \mathrm{Homeo}(X)$ to be continuous with respect to the compact-open topology on $\mathrm{Homeo}(X)$, or some other topology, depending on what X is (for example, one could take the C^∞ topology on $\mathrm{Diff}(X)$ if X is a smooth manifold). Also note that we have defined a *left* action of G on X. There is a corresponding notion of right G-action $(x,g) \mapsto x \cdot g$. For example, one usually takes $\pi_1 X$ to act on the right by covering transformations on the universal cover of X.

4.2. Fiber bundles

We can now give a definition of fiber bundles.

Definition 4.2. Let G be a topological group acting effectively on a space F. A *fiber bundle E over B with fiber F and structure group G* is a map $p : E \to B$ together with a collection of homeomorphisms $\{\varphi : U \times F \to p^{-1}(U)\}$ for open sets U in B (φ is called a *chart over* U) such that:

1. The diagram
$$\begin{tikzcd} U \times F \arrow[rr, "\varphi"] \arrow[dr, "p_U"'] & & p^{-1}(U) \arrow[dl, "p"] \\ & U & \end{tikzcd}$$
commutes for each chart φ over U.
2. Each point of B has a neighborhood over which there is a chart.
3. If φ is a chart over U and $V \subset U$ is open, then the restriction of φ to V is a chart over V.
4. For any charts φ, φ' over U, there is a continuous map $\theta_{\varphi,\varphi'} : U \to G$ so that
$$\varphi'(u,f) = \varphi(u, \theta_{\varphi,\varphi'}(u) \cdot f)$$
for all $u \in U$ and all $f \in F$. The map $\theta_{\varphi,\varphi'}$ is called the *transition function* for φ, φ'.
5. The collection of charts is maximal among collections satisfying the previous conditions.

The standard terminology is to to call B the *base*, F is called the *fiber*, and E is called the *total space*. For shorthand one often abbreviates (p, E, B, F) by E.

This definition of fiber bundle is slick and some discussion about the various requirements helps to understand the concept.

A map $p : E \to B$ is called a *locally trivial bundle* if the first 3 requirements of Definition 4.2 are met. There is no need to assume that any group G is acting since this does not enter into the first three axioms. Local triviality is the important distinction between a fiber bundle and an arbitrary map.

The fourth condition invokes the structure group G. To understand the difference between a locally trivial bundle and a fiber bundle, notice that in a locally trivial bundle, if

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi"] \arrow[dr, "p_U"'] & & p^{-1}(U) \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

and

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi'"] \arrow[dr, "p_U"'] & & p^{-1}(U) \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

are two local trivializations, then commutativity of the diagram

$$\begin{tikzcd} U \times F \arrow[r, "\varphi'"] \arrow[dr, "p_U"'] & p^{-1}(U) \arrow[r, "\varphi^{-1}"] \arrow[d, "p"] & U \times F \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

implies that there is a map $\psi_{\varphi,\varphi'} : U \times F \to F$ so that the composite $\varphi^{-1} \circ \varphi' : U \times F \to U \times F$ has the formula

$$(u, f) \mapsto (u, \psi_{\varphi,\varphi'}(u, f)).$$

For each $u \in U$ the map $\psi_{\varphi,\varphi'}(u, -) : F \to F$ is a homeomorphism.

In a fiber bundle, the map $\psi_{\varphi,\varphi'}$ must have a very special form, namely

1. The homeomorphism $\psi_{\varphi,\varphi'}(u, -) : F \to F$ is not arbitrary, but is given by the action of an element of G, i.e. $\psi_{\varphi,\varphi'}(u, f) = g \cdot f$ for some $g \in G$ independent of f. The element g is denoted by $\theta_{\varphi,\varphi'}(u)$.
2. The topology of G is integrated into the structure by requiring that $\theta_{\varphi,\varphi'} : U \to G$ be continuous.

The requirement that G act effectively on F implies that the functions $\theta_{\varphi,\varphi'} : U \to G$ are unique. Although we have included the requirement that G acts effectively of F in the definition of a fiber bundle, there are certain circumstances when we will want to relax this condition, particularly when studying liftings of the structure group, for example, when studying local coefficients.

It is not hard to see that a locally trivial bundle is the same thing as a fiber bundle with structure group $\mathrm{Homeo}(F)$. One subtlety about the topology is that the requirement that G be a topological group acting effectively on F says only that the homomorphism $G \to \mathrm{Homeo}(F)$ is injective, but the inclusion $G \to \mathrm{Homeo}(F)$ need not be an embedding, nor even continuous.

Exercise 56. Show that the transition functions determine the bundle. That is, suppose that spaces B and F are given, and an action of a topological group G on F is specified.

Suppose also that a collection of pairs $\mathcal{T} = \{(U_\alpha, \theta_\alpha)\}$ with each U_α an open subset of B and $\theta_\alpha : U_\alpha \to G$ a continuous map is given satisfying:

1. The U_α cover B.
2. If $(U_\alpha, \theta_\alpha) \in \mathcal{T}$ and $W \subset U_\alpha$, then the restriction $(W, \theta_{\alpha|W})$ is in the collection $\mathcal{T}$.
3. If (U, θ_1) and (U, θ_2) are in $\mathcal{T}$, then $(U, \theta_1 \cdot \theta_2)$ is in $\mathcal{T}$, where $\theta_1 \cdot \theta_2$ means the pointwise multiplication of functions to G.
4. the collection $\mathcal{T}$ is maximal with respect to the first three conditions.

Then there exists a fiber bundle $p : E \to B$ with structure group G, fiber F, and transition functions θ_α.

The third condition in Exercise 56 is a hidden form of the famous "cocycle condition". Briefly what this means is the following. In an alternative definition of a fiber bundle one starts with a fixed open cover $\{U_i\}$ and a single function $\phi_i : U_i \times F \to p^{-1}(U_i)$ of each open set U_i of the cover. Then to each pair of open sets U_i, U_j in the cover one requires there exists a function $\theta_{i,j} : U_i \cap U_j \to G$ so that (on $U_i \cap U_j$)

$$\phi_i^{-1} \circ \phi_j(u, f) = (u, \theta_{i,j}(u) \cdot f).$$

A *G-valued Čech 1-cochain for the cover* $\{U_i\}$ is just a collection of maps $\theta_{i,j} : U_i \cap U_j \to G$ and so a fiber bundle with structure group G determines a Čech 1-cochain.

From this point of view the third condition of the exercise translates into the requirement that for each triple U_i, U_j and U_k the restrictions of the various θ satisfy

$$\theta_{i,j} \cdot \theta_{j,k} = \theta_{i,k} : U_i \cap U_j \cap U_k \to G.$$

In the Čech complex this condition is just the requirement that the Čech 1-cochain defined by the $\theta_{i,j}$ is in fact a cocycle.

This is a useful method of understanding bundles since it relates them to (Čech) cohomology. Cohomologous cochains define isomorphic bundles, and so equivalence classes of bundles over B with structure group G can be identified with $H^1(B;G)$ (this is one starting point for the theory of characteristic classes; we will take a different point of view in a later chapter). One must be extremely cautious when working this out carefully. For example, G need not be abelian (and so what does $H^1(B;G)$ mean?) Also, one must consider *continuous cocycles* since the $\theta_{i,j}$ should be continuous functions. We will not pursue this line of exposition any further in this book.

We will frequently use the notation $F \hookrightarrow E \xrightarrow{p} B$ or

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

to indicate a fiber bundle $p : E \to B$ with fiber F.

4.3. Examples of fiber bundles

The following are some examples of locally trivial bundles. We will revisit these and many more examples in greater detail in Section 6.14.

1. The *trivial* bundle is the projection $p_B : B \times F \to B$.
2. If F has the discrete topology, any locally trivial bundle over B with fiber F is a covering space; conversely if $p : E \to B$ is a covering space with B connected, then p is a locally trivial bundle with discrete fiber.
3. The Möbius strip mapping onto its core circle is a locally trivial bundle with fiber $[0, 1]$.
4. The tangent bundle of a smooth manifold is a locally trivial bundle.

Exercise 57. Show that a fiber bundle with with trivial structure group is (isomorphic to) a trivial bundle.

4.3.1. Vector bundles.

Exercise 58. Let $F = \mathbf{R}^n$, and let $G = GL_n\mathbf{R} \subset \text{Homeo}(\mathbf{R}^n)$. A fiber bundle over B with fiber $\mathbf{R}^n$ and structure group $GL_n(\mathbf{R})$ is a *vector bundle* of *dimension n over B*. Explicitly, show that each fiber $p^{-1}\{b\}$ can be given a well-defined vector space structure.

(Similarly, one can take $F = \mathbf{C}^n$, $G = GL_n(\mathbf{C})$ to get *complex vector bundles*.)

In particular, if M is a differentiable n-manifold, and TM is the set of all tangent vectors to M then $p : TM \to M$ is a vector bundle of dimension n.

4.3.2. Bundles over S^2. For every integer $n \geq 0$, we can construct an S^1 bundle over S^2 with structure group $SO(2)$; n is called the *Euler number* of the bundle. For $n = 0$, we have the product bundle $p : S^2 \times S^1 \to S^2$. For $n \geq 1$, define a 3-dimensional lens space $L_n^3 = S^3/\mathbf{Z}_n$, where the action is given by letting the generator of $\mathbf{Z}_n$ on act on $S^3 \subset \mathbf{C}^2$ by $(z_1, z_2) \mapsto (\zeta_n z_1, \zeta_n z_2)$ (here $\zeta_n = \exp^{2\pi i/n}$ is a primitive n-th root of unity). For $n = 2$, the lens space is just real projective space $\mathbf{R}P^3$. Define the S^1-bundle with Euler number $n \geq 1$ by $p : L_n^3 \to S^2 = \mathbf{C} \cup \infty$ by $[z_1, z_2] \to z_1/z_2$.

When $n = 1$ we obtain the famous Hopf bundle $S^1 \hookrightarrow S^3 \to S^2$. For $n > 1$ the Hopf map $S^3 \to S^2$ factors through the quotient map $S^3 \to L_n^3$, and the fibers of the bundle with Euler number n are $S^1/\mathbf{Z}_n$ which is again homeomorphic to S^1.

Exercise 59. Let $S(TS^2)$ be the sphere bundle of the tangent bundle of the 2-sphere, i.e. the tangent vectors of unit length, specifically

$$S(TS^2) = \{(P, v) \in \mathbf{R}^3 \times \mathbf{R}^3 | P, v \in S^2 \text{ and } P \cdot v = 0\}.$$

Let $SO(3)$ be the 3-by-3 orthogonal matrices of determinant one (the group of orientation preserving rigid motions of $\mathbf{R}^3$ preserving the origin). This is a topological group. Show that the spaces $S(TS^2)$, $SO(3)$, and $\mathbf{R}P^3$ are all homeomorphic.

(Hint.

1. Given two perpendicular vectors in $\mathbf{R}^3$, a third one can be obtained by the cross product.
2. On one hand, every element of $SO(3)$ is rotation about an axis, on the other hand $\mathbf{R}P^3$ is $D^3/\sim$, where you identify antipodal points on the boundary sphere.)

This gives three incarnations of the S^1-bundle over S^2 with Euler number equal to 2:

1. $p : S(TS^2) \to S^2$, $(P, v) \mapsto P$
2. $p : SO(3) \to S^2$, $A \mapsto A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
3. $p : \mathbf{R}P^3 \to S^2$, the lens space bundle above.

4.3.3. Clutching. Suppose a topological group G acts on a space F. Let X be a space and let ΣX be the unreduced suspension of X,

$$\Sigma X = \frac{X \times I}{(x, 0) \sim (x', 0), (x, 1) \sim (x', 1)}.$$

Then given a map $\beta : X \to G$, define

$$E = \frac{(X \times [0,1/2] \times F) \amalg (X \times [1/2,1] \times F)}{\sim}$$

where the equivalence relation is given by identifying $(x,0,f) \sim (x',0,f)$, $(x,1,f) \sim (x',1,f)$, and $(x,1/2,f) \sim (x,1/2,\beta(x)f)$, where the last relation glues the summands of the disjoint union. This bundle is called the *bundle over ΣX with clutching function $\beta : X \to G \subset Aut(F)$.*

Exercise 60. Show that projection onto the first two coordinates gives a fiber bundle $p : E \to \Sigma X$ with fiber F and structure group G. Give some examples with $X = S^0$ and $X = S^1$. In particular, show that the S^1-bundle over $S^2 = \Sigma S^1$ with Euler number equal to n is obtained by clutching using a degree n map $S^1 \to S^1$.

Clutching provides a good way to describe fiber bundles over spheres. For X a CW-complex, all bundles over ΣX arise by this clutching construction. This follows from the fact that any fiber bundle over a contractible CW-complex is trivial (this can be proven using obstruction theory). Since ΣX is the union of two contractible spaces, $X \times [0, \frac{1}{2}]/\sim$ and $X \times [\frac{1}{2}, 1]/\sim$, any bundle over ΣX is obtained by clutching two trivial bundles over X.

4.3.4. Local coefficients and other structures. An important type of fiber bundle is the following. Let A be a group and G a subgroup of the automorphism group $\mathrm{Aut}(A)$. Then any fiber bundle E over B with fiber A and structure group G has the property that each *fiber* $p^{-1}\{b\}$ has a group structure. This group is isomorphic to A, but the isomorphism is not canonical in general.

We have already run across an important case of this, namely vector bundles, where $A = \mathbf{R}^n$ and $G = GL_n(\mathbf{R})$.

In particular, if A is a abelian group with the discrete topology, then $p : E \to B$ is a covering space and is called a *system of local coefficients on* B. The terminology will be explained later.

Exercise 61. Define local coefficient systems for R-modules, R a commutative ring, generalizing the case of $\mathbf{Z}$-modules above.

The basic principle at play here is *if the structure group preserves a certain structure on F, then every fiber $p^{-1}\{b\}$ has this structure.* For example, a local coefficient system corresponds to the case when the structure group is a subgroup of the group of automorphisms of the fiber, a discrete abelian group. A vector bundle corresponds to the case when the structure group corresponds to the group of linear transformations of a vector space. Other examples of fibers with a structure that one could consider include the following.

1. F is a real vector space with an inner product, $G = O(F, \langle\ ,\ \rangle) \subset GL(F)$ consists of those linear isomorphisms which preserve the inner product. The resulting fiber bundle is called a *vector bundle with an orthogonal structure.*
2. Similarly one can define a *complex vector bundle with hermitian structure* by taking F to be a complex vector space with a hermitian inner product.
3. Taking this further, let F be a riemannian manifold and suppose that G acts isometrically on F. Then each fiber in a fiber bundle with structure group G and fiber F will be (non-canonically) isometric to F.
4. Take F to be a smooth manifold and G a subgroup of the diffeomorphism group of F (with the C^∞ strong topology, say). Then each fiber in a fiber bundle with structure group G will be diffeomorphic to F.

Exercise 62. Invent your own examples of fibers with structure and the corresponding fiber bundles.

4.4. Principal bundles and associated bundles

Principal bundles are special cases of fiber bundles, but nevertheless can be used to construct any fiber bundle. Conversely any fiber bundle determines a principal bundle. A principal bundle is technically simpler, since the fiber is just $F = G$ with a canonical action.

Let G be a topological group. It acts on itself by *left translation.*

$$G \to \mathrm{Homeo}(G), \quad g \mapsto (x \mapsto gx).$$

Definition 4.3. A *principal G-bundle over B* is a fiber bundle $p : P \to B$ with fiber $F = G$ and structure group G acting by left translations.

Proposition 4.4. *If $p : P \to B$ is a principal G-bundle, then G acts freely on P on the right with orbit space B.*

Proof. Notice first that G acts on the local trivializations *on the right*:

$$(U \times G) \times G \to U \times G$$
$$(u, g) \cdot g' = (u, gg').$$

This commutes with the action of G on itself by left translation (i.e. $(g''g)g' = g''(gg')$), so one gets a well-defined right action of G on E using the identification provided by a chart

$$U \times G \xrightarrow{\varphi} p^{-1}(U).$$

More explicitly, define $\varphi(u,g)\cdot g' = \varphi(u,gg')$. If φ' is another chart over U, then

$$\varphi(u,g) = \varphi'(u,\theta_{\varphi,\varphi'}(u)g),$$

and $\varphi(u,gg') = \varphi'(u,\theta_{\varphi,\varphi'}(u)(gg')) = \varphi'(u,(\theta_{\varphi,\varphi'}(u)g)g')$, so the action in independent of the choice of chart. The action is free, since the local action $(U\times G)\times G \to U\times G$ is free, and since $U\times G/G = U$ it follows that $E/G = B$. □

As a familiar example, any regular covering space $p : E \to B$ is a principal G-bundle with $G = \pi_1 B/p_*\pi_1 E$. Here G is given the discrete topology. In particular, the universal covering $\tilde{B} \to B$ of a space is a principal $\pi_1 B$-bundle. A non-regular covering space is not a principal G-bundle.

Exercise 63. Any free (right) action of a finite group G on a (Hausdorff) space E gives a regular cover and hence a principal G-bundle $E \to E/G$.

The converse to Proposition 4.4 holds in some important cases. We state the following fundamental theorem without proof, referring you to [**5**, Theorem II.5.8].

Theorem 4.5. *Suppose that X is a compact Hausdorff space, and G is a compact Lie group acting freely on X. Then the orbit map*

$$X \to X/G$$

is a principal G-bundle. □

4.4.1. Construction of fiber bundles from principal bundles. Exercise 56 shows that the transition functions $\theta_\alpha : U_\alpha \to G$ and the action of G on F determine a fiber bundle over B with fiber F and structure group G.

As an application note that if a topological group G acts on spaces F and F', and if $p : E \to B$ is a fiber bundle with fiber F and structure group G, then one can use the transition functions from p to define a fiber bundle $p' : E' \to B$ with fiber F' and structure group G with *exactly the same transition functions.*

This is called *changing the fiber from F to F'.* This can be useful because the topology of E and E' may change. For example, take $G = GL_2(\mathbf{R})$, $F = \mathbf{R}^2$, $F' = \mathbf{R}^2 - \{0\}$ and the tangent bundle of the 2-sphere.

$$\begin{tikzcd} \mathbf{R}^2 \arrow[r] & TS^2 \arrow[d, "p"] \\ & S^2 \end{tikzcd}$$

After changing the fiber from $\mathbf{R}^2$ to $\mathbf{R}^2 - \{0\}$ we obtain

$$\begin{tikzcd} \mathbf{R}^2 - \{0\} \arrow[r] & TS^2 - z(S^2) \arrow[d, "p"] \\ & S^2 \end{tikzcd}$$

where $z : S^2 \to TS^2$ denotes the zero section.

With the second incarnation of the bundle the twisting becomes revealed in the homotopy type, because the total space of the first bundle has the homotopy type of S^2, while the total space of the second has the homotopy type of the sphere bundle $S(TS^2)$ and hence of $\mathbf{R}P^3$ according to Exercise 59.

A fundamental case of changing fibers occurs when one lets the fiber F' be the group G itself, with the left translation action. Then the transition functions for the fiber bundle

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

determine, via the construction of Exercise 56, a principal G-bundle

$$\begin{tikzcd} G \arrow[r] & P(E) \arrow[d, "p"] \\ & B. \end{tikzcd}$$

We call this principal G-bundle the *principal G-bundle underlying the fiber bundle $p : E \to B$ with structure group G.*

Conversely, to a principal G-bundle and an action of G on a space F one can associate a fiber bundle, again using Exercise 56. An alternative construction is given in the following definition.

Definition 4.6. Let $p : P \to B$ be a principal G-bundle. Suppose G acts on the left on a space F, i.e. an action $G \times F \to F$ is given. Define the *Borel construction*

$$P \times_G F$$

to be the quotient space $P \times F/\sim$ where

$$(x, f) \sim (xg, g^{-1}f).$$

(We are continuing to assume that G acts on F on the left and by Proposition 4.4 it acts freely on the principal bundle P on the right).

Let $[x, f] \in P \times_G F$ denote the equivalence classes of (x, f). Define a map

$$q : P \times_G F \to B$$

by the formula $[x, f] \mapsto p(x)$.

The following important exercise shows that the two ways of going from a principal G-bundle to a fiber bundle with fiber F and structure group G are the same.

Exercise 64. If $p : P \to B$ is a principal G-bundle and G acts on F, then

$$\begin{tikzcd} F \arrow[r] & P \times_G F \arrow[d, "q"'] \\ & B \end{tikzcd}$$

where $q[x, f] = p(x)$, is a fiber bundle over B with fiber F and structure group G which has the same transition functions as $p : P \to B$.

We say $q : E \times_G F \to B$ is *the fiber bundle associated to the principal bundle $p : E \to B$ via the action of G on F.*

Thus principal bundles are more basic that fiber bundles, in the sense that the fiber and its G-action are explicit, namely G acting on itself by left translation. Moreover, any fiber bundle with structure group G is associated to a principal G-bundle by specifying an action of G on a space F. Many properties of bundles become more visible when stated in the context of principal bundles.

The following exercise gives a different method of constructing the principal bundle underlying a vector bundle, without using transition functions.

Exercise 65. Let $p : E \to B$ be a vector bundle with fiber $\mathbf{R}^n$ and structure group $GL(n, \mathbf{R})$. Define a space $F(E)$ to be the space of *frames* in E, so that a point in $F(E)$ is a pair $(b, \mathbf{f})$ where $b \in B$ and $\mathbf{f} = (f_1, \cdots, f_n)$ is a basis for the vector space $p^{-1}(b)$. There is an obvious map $q : F(E) \to B$.

Prove that $q : F(E) \to B$ is a principal $GL(n, \mathbf{R})$-bundle, and that

$$E = F(E) \times_{GL(n,\mathbf{R})} \mathbf{R}^n$$

where $GL(n, \mathbf{R})$ acts on $\mathbf{R}^n$ in the usual way.

For example, given a representation of $GL(n, \mathbf{R})$, that is, a homomorphism $\rho : GL(n, \mathbf{R}) \to GL(k, \mathbf{R})$, one can form a new vector bundle

$$F(E) \times_\rho \mathbf{R}^k$$

over B.

An important set of examples comes from this construction by starting with the tangent bundle of a smooth manifold M. The principal bundle

$F(TM)$ is called the *frame bundle* of M. Any representation of $GL(n, \mathbf{R})$ on a vector space V gives a vector bundle with fiber isomorphic to V. Important representations include the *alternating representations* $GL(n, \mathbf{R}) \to \bigwedge^p(\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}))$ from which one obtains the vector bundles of differential p-forms over M.

We next give one application of the Borel construction. Recall that a local coefficient system is a fiber bundle over B with fiber A and structure group G where A is a (discrete) abelian group and G acts via a homomorphism $G \to \mathrm{Aut}(A)$.

Lemma 4.7. *Every local coefficient system over a path-connected (and semi-locally simply connected) space B is of the form*

$$\begin{array}{ccc} A & \longrightarrow & \tilde{B} \times_{\pi_1 B} A \\ & & \downarrow{\scriptstyle q} \\ & & B \end{array}$$

i.e., is associated to the principal $\pi_1 B$-bundle given by the universal cover $\tilde{B}$ of B where the action is given by a homomorphism $\pi_1 B \to Aut(A)$.

In other words the group $G \subset \mathrm{Aut}(A)$ can be replaced by the discrete group $\pi_1 B$. Notice that in general one cannot assume that the homomorphism $\pi_1 B \to \mathrm{Aut}(A)$ is injective, and so this is a point where we would wish to relax the requirement that the structure group acts effectively on the fiber. Alternatively, one can take the structure group to be $\pi_1(B)/\ker(\phi)$ where $\phi : \pi_1(B) \to \mathrm{Aut}(A)$ is the corresponding representation.

Sketch of proof. It is easy to check that $q : \tilde{B} \times_{\pi_1 B} A \to B$ is a local coefficient system, i.e. a fiber bundle with fiber on abelian group A and structure group mapping to $\mathrm{Aut}(A)$.

Suppose that $p : E \to B$ is any local coefficient system. Any loop $\gamma : (I, \partial I) \to (B, *)$ has a unique lift to E starting at a given point in $p^{-1}(*)$, since A is discrete so that $E \to B$ is a covering space. Fix an identification of $p^{-1}(*)$ with A, given by a chart. Then the various lifts of γ starting at points of A define, by taking the end point, a function $A \to A$.

The fact that $p : E \to B$ has structure group $\mathrm{Aut}(A)$ easily implies that this function is an automorphism. Since E is a covering space of B, the function only depends on the homotopy class, and so we get a map $\pi_1(B, *) \to \mathrm{Aut}(A)$. This is clearly a homomorphism since if $\tilde{\gamma_1}, \tilde{\gamma_2}$ are lifts starting at a, b, then $\tilde{\gamma_1} + \tilde{\gamma_2}$ (addition in A) is the lift of $\gamma_1\gamma_2$ (multiplication in π_1) and starts at $a + b$. A standard covering space argument implies that $E = \tilde{B} \times_{\pi_1 B} A$. $\square$

4.5. Reducing the structure group

In some circumstances, given a subgroup H of G and a fiber bundle $p : E \to B$ with structure group G, one can view the bundle as a fiber bundle with structure group H. When this is possible, we say *the structure group can be reduced to H*.

Proposition 4.8. *Let H be a topological subgroup of the topological group G. Let H act on G by left translation. Let $q : Q \to B$ be a principal H-bundle. Then*

$$\begin{tikzcd} G \arrow[r] & Q \times_H G \arrow[d, "q"] \\ & B \end{tikzcd}$$

is a principal G-bundle. □

The proof is easy; one approach is to consider the transition functions $\theta : U \to H$ as functions to G using the inclusion $H \subset G$. To satisfy maximality of the charts it may be necessary to add extra charts whose transition functions into G map outside of H.

Exercise 66. Prove Proposition 4.8.

Definition 4.9. Given a principal G-bundle $p : E \to B$ we say *the structure group G can be reduced to H* for some subgroup $H \subset G$ if there exists a principal H-bundle $Q \to B$ and a commutative diagram

$$\begin{tikzcd} Q \times_H G \arrow[rr, "r"] \arrow[dr] & & E \arrow[dl] \\ & B & \end{tikzcd}$$

so that the map r is G-equivariant. For a fiber bundle, we say the structure group reduces if the structure group of the underlying principal bundle reduces.

If we are willing to relax the requirement that the structure group acts effectively, then we can just assume that we are given a homomorphism $H \to G$ rather than an inclusion of a subgroup. Proposition 4.8 holds without change. In this more general context, for example, Lemma 4.7 states that any fiber bundle over B with discrete fiber can have its structure group reduced to $\pi_1 B$.

Exercise 67. Show that every real vector bundle (i.e. fiber bundle with structure group $GL(n, \mathbf{R})$ acting on $\mathbf{R}^n$ in the usual way) over a paracompact base can have its structure group reduced to the orthogonal group $O(n)$. (Hint: use a partition of unity.)

Another subtle point is that there may be several "inequivalent" reductions. An example concerns orientability and orientation of vector bundles.

Definition 4.10. A real vector bundle is called *orientable* if its structure group can be reduced to the subgroup $GL_+(n, \mathbf{R})$ of matrices with positive determinant.

For example, a smooth manifold is orientable if and only if its tangent bundle is orientable. A more detailed discussion of orientability for manifolds and vector bundles can be found in Section 10.7.

For the following exercise it may help to read the definition of a map between fiber bundles in the next section.

Exercise 68. Prove that an orientable vector bundle can be oriented in two incompatible ways, that is, the structure group can be reduced from $GL(n, \mathbf{R})$ to $GL_+(n, \mathbf{R})$ (or, using Exercise 67, from $O(n)$ to $SO(n)$) in two ways so that the identity map Id: $E \to E$ is a not a map of fiber bundles with structure group $GL_+(n, \mathbf{R})$ (or $SO(n)$).

4.6. Maps of bundles and pullbacks

The concept of morphisms of fiber bundles is subtle, especially when there are different fibers and structure groups. Rather than to try to work in the greatest generality, we will just define one of many possible notions of morphism.

Definition 4.11. A *morphism of fiber bundles with structure group G and fiber F* from $E \to B$ to $E' \to B'$ is a pair of continuous maps $\tilde{f} : E \to E'$ and $f : B \to B'$ so that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

commutes and so that for each chart $\phi : U \times F \to p^{-1}(U)$ with $b \in U$ and chart $\phi' : U' \times F \to p^{-1}(U')$ and each $b \in U$ with $f(b) \in U'$ the composite

$$\{b\} \times F \xrightarrow{\phi} p^{-1}(b) \xrightarrow{\tilde{f}} (p')^{-1}(f(b)) \xrightarrow{(\phi')^{-1}} \{f(b)\} \times F$$

is a homeomorphism given by the action of an element $\psi_{\phi,\phi'}(b) \in G$. Moreover, $b \mapsto \psi_{\phi,\phi'}(b)$ should define a continuous map from $U \cap f^{-1}(U')$ to G.

As you can see, this is a technical definition. Notice that the fibers are mapped homeomorphically by a map of fiber bundles of this type. In particular, an *isomorphism* of fiber bundles is a map of fiber bundles $(\tilde{f}, f)$ which admits a map $(\tilde{g}, g)$ in the reverse direction so that both composites are the identity.

One important type of fiber bundle map is a *gauge transformation*. This is a bundle map from a bundle to itself which covers the identity map of the base, i.e. the following diagram commutes.

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

By definition g restricts to an isomorphism given by the action of an element of the structure group on each fiber. The set of all gauge transformations forms a group.

One way in which morphisms of fiber bundles arise is from a pullback construction.

Definition 4.12. Suppose that a fiber bundle $p : E \to B$ with fiber F and structure group G is given, and that $f : B' \to B$ is some continuous function. Define the *pullback of* $p : E \to B$ *by* f to be the space

$$f^*(E) = \{(b', e) \in B' \times E \mid p(e) = f(b')\}.$$

Let $q : f^*(E) \to B$ be the restriction of the projection $E \times B \to B$ to $f^*(E)$. Notice that there is a commutative diagram

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

Theorem 4.13. *The map $q : f^*(E) \to B'$ is a fiber bundle with fiber F and structure group G. The map $f^*(E) \to E$ is a map of fiber bundles.*

Proof. This is not hard. The important observation is that if φ is a chart over $U \subset B$, then $f^{-1}(U)$ is open in B' and φ induces a homeomorphism $f^{-1}(U) \times F \to f^*(E)_{|f^{-1}(U)}$. We leave the details as an exercise. $\square$

The following exercise shows that any map of fiber bundles is given by a pullback.

Exercise 69. Let

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

be a map of bundles with fiber F in the sense of Definition 4.11. Show that there is a factorization

$$\begin{tikzcd} E' \arrow[r, dotted, "\beta"] \arrow[dr, "p'"'] & f^*E \arrow[r, "f^*"] \arrow[d, "q"] & E \arrow[d, "p"] \\ & B' \arrow[r, "f"'] & B \end{tikzcd}$$

so that $f^* \circ \beta = \tilde{f}$, with (β, Id) a map of bundles over B'.

We have given a rather narrow and rigid definition of fiber bundle morphisms. More general definitions can be given depending on the structure group, fiber, etc..

Exercise 70. Define a morphism between two fiber bundles with structure group G but with different fibers by requiring the map on fibers to be equivariant. Use this to define a morphism of vector bundles.

4.7. Projects for Chapter 4

4.7.1. Fiber bundles over paracompact bases are fibrations. State and prove the theorem of Hurewicz (Theorem 6.8) which says that a map $f : E \to B$ with B paracompact is a fibration (see Definition 6.7) provided that B has an open cover $\{U_i\}$ so that $f : f^{-1}(U_i) \to U_i$ is a fibration for each i. In particular, any locally trivial bundle over a paracompact space is a fibration.

A reference for the proof is [**10**, Chapter XX,§3-4] or [**36**].

4.7.2. Classifying spaces. For any topological group G there is a space BG and a principal G-bundle $EG \to BG$ so that given any paracompact space B, the pullback construction induces a bijection between the set $[B, BG]$ of homotopy classes of maps from B to BG and isomorphism classes of principal G-bundles over B. Explain the construction of the bundle $EG \to BG$ and prove this theorem. Show that the assignment $G \mapsto BG$ is functorial with respect to continuous homomorphisms of topological groups. Show that a principal G-bundle P is of the form $Q \times_H G$ (as in Proposition

4.8) if and only if the classifying map $f : B \to BG$ lifts to BH

$$\begin{tikzcd} & BH \arrow[d] \\ B \arrow[ur, dotted] \arrow[r, "f"'] & BG \end{tikzcd}$$

Show that given any action of G on F, any fiber bundle $E \to B$ with structure group G and fiber F is isomorphic to the pullback

$$f^*(EG \times_G F)$$

where $f : B \to BG$ classifies the principal G-bundle underlying $E \to B$. Use this theorem to define characteristic classes for principal bundles.

See Theorem 8.22 and Corollary 6.50 for more on this important topic.

A reference for this material is [**17**]. We will use these basic facts about classifying spaces throughout this book, notably when we study bordism.