SECTION 9: CELLULAR APPROXIMATION

The aim of this section is to give a proof of the **cellular approximation theorem**. One formulation of this theorem is that every map between CW complexes is homotopic to a cellular map (but we will also see a version for CW pairs). Recall that a map of CW complexes is cellular if it restricts to a map of n-skeleta for all n. Thus, such a cellular map does not increase the dimension of cells. The proof of this theorem is rather involved and will occupy the bulk of this section. The main work will be to establish Lemma 4 which covers a particular case. In the next section we will use the cellular approximation theorem in order to deduce the famous 'Whitehead's theorem'.

Before we begin with the cellular approximation theorem, let us recall a fact from point-set topology. In general, it is not true that the formation of quotient spaces and products would be compatible. More precisely, let $q\colon X \to Y$ be a map exhibiting Y as a quotient of X. In particular, a subset of Y is open if and only if the preimage $q^{-1}(U)$ is open in X. If Z is an arbitrary space, then, in general, we can not conclude that the map $q \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is a quotient map. However, there is the following fact.

Lemma 1. *Let $q\colon X \to Y$ be a quotient map and let K be a compact space. Then also the map $q \times \mathrm{id}_K\colon X \times K \to Y \times K$ is a quotient map.*

We will be particularly interested in the following situation. Let X be a CW complex. The n-skeleton $X^{(n)}$ of X is obtained from the $(n-1)$-skeleton by attaching a set of n-cells. Thus, we have a quotient map

$$q_n\colon X^{(n-1)} \sqcup J_n \times e^n \to X^{(n)}.$$

If we form the product of this map with the identity of $I = [0,1]$ then the previous lemma implies the following.

Corollary 2. *Let X be a CW complex. Then there is a quotient map*

$$q_n \times \mathrm{id}_I\colon X^{(n-1)} \times I \sqcup J_n \times e^n \times I \to X^{(n)} \times I.$$

Let us immediately give the statement of the cellular approximation theorem.

Theorem 3. *Let (X, A) be a CW pair, let Y be a CW complexes, and let $f\colon X \to Y$ be a map of spaces. If $f|_A\colon A \to Y$ is cellular, then f is homotopic to a cellular map $g\colon X \to Y$ relative to A. In particular, any map of CW complexes is homotopic to a cellular one.*

Since CW complexes are built inductively, the following strategy will not come as a surprise. Given a map $f\colon X \to Y$ of CW complexes, we will try to deform f cell by cell into a cellular map. As an important building block for the proof of the theorem, there is the following case of a single cell.

Lemma 4. *Let Y be obtained from B by attaching an n-cell, i.e., assume that we have a pushout diagram of the form*

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & B \arrow[d] \\ D^n \arrow[r, "\chi"'] & Y \end{tikzcd}$$

Then any map $f\colon (D^m, \partial D^m) \to (Y, B)$ with $m < n$ is homotopic relative to ∂D^m to a map g satisfying $g(D^m) \subseteq B$.

Proof. The proof of this lemma will be given at the end of this section. □

Let us now use this lemma to give a proof of the cellular approximation theorem. As usual, given a CW complex Y, the skeleton filtration will be denoted by

$$Y^{(0)} \subseteq Y^{(1)} \subseteq Y^{(2)} \subseteq \ldots, \qquad n \in \mathbb{N}.$$

Proof. (**of Theorem 3 (using Lemma 4)**) Thus, we are given a map $g_{-1} = f\colon X \to Y$ which restricts to a cellular map on $A = X^{(-1)}$. We will inductively construct maps $g_n\colon X \to Y$ and homotopies $H_n\colon g_{n-1} \simeq g_n$ for $n \geq 0$ such that

(1) The map g_n sends the relative n-cells,i.e., the ones given by the index set $J_n(X) - J_n(A)$, to $Y^{(n)}$.
(2) The homotopy $H_n\colon g_{n-1} \simeq g_n$ is relative to $X^{(n-1)}$.

Recall that a CW pair is, in particular, a relative CW complex so that we have a filtration

$$A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X,$$

which has the following two properties:

(1) The space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells for $n \geq 0$.
(2) The space X is the union $\bigcup_{n \geq -1} X^{(n)}$ endowed with the weak topology.

Let us assume inductively that we are already given the map g_{n-1}, and let us construct g_n and H_n. Now, denoting the set of relative n-cells by J_n, there is a pushout diagram

$$\begin{tikzcd}
J_n \times \partial e^n = \coprod_{\sigma \in J_n} \partial e^n_\sigma \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\
J_n \times e^n = \coprod_{\sigma \in J_n} e^n_\sigma \arrow[r, "(\chi_\sigma)_\sigma"'] & X^{(n)}.
\end{tikzcd}$$

Let us assume that there are cells $\sigma \in J_n$ such that $g_{n-1}(e^n_\sigma)$ is not contained in $Y^{(n)}$ (otherwise we set $g_n = g_{n-1}$ and take H_n to be the constant homotopy). For each such cell e^n_σ, there is a *finite relative* subcomplex Y' with $Y^{(n)} \subseteq Y' \subseteq Y$ such that $g_{n-1}(e^n_\sigma) \subseteq Y'$. Take a cell of maximal dimension in Y' which has a nontrivial intersection with $g_{n-1}(e^n_\sigma)$. Then Lemma 4 tells us that this cell can be avoided up to relative homotopy. Repeating this finitely many times and gluing the relative homotopies together, we obtain a homotopy $H_{n,\sigma}\colon g_{n-1} \simeq g_{n,\sigma}\colon e^n_\sigma \to Y$ relative to ∂e^n_σ such that $g_{n,\sigma}(e^n_\sigma) \subseteq Y^{(n)}$. Recall from Corollary 2 that $X^{(n)} \times [0,1]$ carries the quotient topology with respect to the map

$$X^{(n-1)} \times [0,1] \sqcup J_n \times e^n \times [0,1] \to X^{(n)} \times [0,1].$$

Thus, we can glue the homotopies $H_{n,\sigma}$, the constant homotopies on $g_{n-1}\colon e^n_\sigma \to Y$ for all n-cells with $g_{n-1}(e^n_\sigma) \subseteq Y^{(n)}$, and the constant homotopy on $g_{n-1}|_{X^{(n-1)}}$ together in order to obtain a homotopy

$$\widetilde{H}_n\colon g_{n-1}|_{X^{(n)}} \simeq \widetilde{g}_n\colon X^{(n)} \times [0,1] \to Y.$$

From these data we can form the following extension problem

$$\begin{tikzcd} X \times \{0\} \cup X^{(n)} \times [0,1] \arrow[r, "(g_{n-1},\widetilde{H}_n)"] \arrow[d] & Y \\ X \times [0,1] \arrow[ur, dashed, "\exists H_n"'] & \end{tikzcd}$$

which admits a solution since the inclusion $X^{(n)} \to X$ is a cofibration. It only remains to set $g_n = H_n(-,1)$ in order to conclude the inductive step.

If the CW complex X is finite-dimensional, i.e., if $X^{(n)} = X$ for some $n \geq 0$, then we are done since it suffices to compose the finitely many homotopies H_k, $0 \leq k \leq n$, to obtain a homotopy $H\colon f \simeq g = g_n$ relative to A such that $g\colon X \to Y$ is a cellular map.

For an infinite-dimensional CW complex we can conclude by the following argument. In that case we have to check that these infinitely many homotopies can be assembled into a single homotopy $H\colon X \times I \to Y$. In fact, as the homotopies H_n are relative to $X^{(n-1)}$, it follows that H_k is stationary on $X^{(n-1)}$ for $k \geq n$. Thus, we define H on $X^{(n-1)}$ by first running through H_0 at a double speed, then through H_1 at a fourfold speed, through H_2 at an 8-fold speed, and so on. After having run through H_{n-1}, the map $H|_{X^{(n-1)}}$ is defined to be stationary. We leave it to the reader to check that this way we obtain a continuous map $H\colon X \times I \to Y$. From the definition it is immediate that H is a homotopy relative to A such that $g = H(-,1)\colon X \to Y$ is a cellular map as intended. □

Corollary 5. *The homotopy groups $\pi_k(S^n,*)$ are trivial for all $1 \leq k < n$.*

Proof. Let us endow S^n with the CW structure consisting of a unique 0-cell and a unique n-cell. By Theorem 3, any map $f\colon S^k \to S^n$ is homotopic to a cellular map $g\colon S^k \to S^n$. But, for $k < n$ we have that the k-skeleton of S^k is the entire k-sphere, while the k-skeleton of S^n consists of a point only. Thus, g is a constant map and we are done. □

We will see more applications of the cellular approximation theorem as the course goes on. In the remainder of this section we give a proof of Lemma 4.

Proof. (**of Lemma 4 (not using Theorem 3)**) The proof will be given by induction over n, the dimension of the cell attached to B. Let us first establish the case of $n = 1$. Thus, $m = 0$ and hence $\partial D^m = \emptyset$ and $D^m = *$. A map $f = \kappa_y\colon (*,\emptyset) \to (Y,B)$ is essentially the same as a point y in Y. There is a path $\omega\colon I \to Y$ with $\omega(0) = y$ and $\omega(1) = b \in B$. This path defines the desired homotopy $f = \kappa_y \simeq \kappa_b = g$.

Before performing the induction step, let us describe the strategy of the proof. The attaching map $\chi\colon D^n \to Y$ restricts to a homeomorphism from the interior of the disc onto its image in Y. The main work consists in showing that we can construct a homotopy $f \simeq h$ relative to the boundary ∂D^m such that h omits the origin of the interior of the attached disc. To see that this is enough, let us denote by $Y - \{o\}$ the space which we obtain from Y by removing that origin. It is easy to see that $i\colon B \to Y - \{o\}$ is the inclusion of a strong deformation retraction (induced by collapsing the punctured n-disc $D^n - \{o\}$ onto S^{n-1}). Part of this strong deformation retraction is a homotopy

$$\mathrm{id}_{Y-\{o\}} \simeq i \circ r \quad \text{relative to } B$$

which induces the desired relative homotopy $h = \mathrm{id} \circ h \simeq i \circ r \circ h = g$ relative to ∂D^m. Putting these two homotopies together we conclude that $f \simeq g$ relative to ∂D^m as intended.

Let us now assume inductively that we already established the lemma for $n-1$. Before attacking the induction step, we list the following consequences of our inductive assumption:

(1) Any map $S^k \to S^{n-1}$ as well as any map $S^k \to S^{n-1} \times (a,b)$ for $k < n-1$ is homotopic to a constant map.
(2) Any map $S^k \to S^{n-1} \times (a,b)$, $k < n-1$, can be extended to a map on D^{k+1}.

In fact, the first case of the first point is established by an application of the lemma for $n-1$ to the map $(D^k, \partial D^k) \to (D^{n-1} \cup_{\partial D^{n-1}} *, *)$ which corresponds to our given map $S^k \to S^{n-1}$. The second case of that point follows immediately from the first one together with the contractibility of the interval. Finally, the second point follows from the fact that a map is homotopic to a constant map if and only if it can be extended over the cone of its domain.

Thus, it remains to construct a homotopy $f \simeq h\colon D^m \to Y$ relative to ∂D^m such that h does not hit the origin. This will be done by a rather elaborate application of the lemma of Lebesgue to an adapted open cover of D^m. We begin by constructing an open cover of Y. For this purpose, let us introduce notations for the subsets

$$U' = \{x \in D^n \mid 0 \leq \|x\| < 2/3\} \qquad \text{and} \qquad V' = \{x \in D^n \mid \|x\| > 1/3\}$$

and let us define two subsets of Y by setting

$$U = \chi(U') \qquad \text{and} \qquad V = B \cup_{\partial D^n} \chi(V')$$

where $\chi\colon D^n \to Y$ is the characteristic map of the n-cell. The aim is to construct a relative homotopy $f \simeq h$ such that the image of h entirely lies in V, and hence, in particular, avoids the point $o \in Y$.

By construction, U and V define an open cover of Y. Since χ induces a homeomorphism when restricted to the interior of the n-disc, we obtain a homeomorphism

$$U \cap V \cong S^{n-1} \times (1/3, 2/3)$$

so that we can later apply the second consequence above to $U \cap V$.

Using our favorite homeomorphism of pairs $(I^m, \partial I^m) \cong (D^m, \partial D^m)$, we are thus in the following situation:

$$\begin{tikzcd}
f\colon & I^m \arrow[r, "\cong"] & D^m \arrow[r, "f"] & Y \\
f\colon & \partial I^m \arrow[u, "\subseteq"] \arrow[r, "\cong"'] & S^{m-1} \arrow[u, "\subseteq"'] \arrow[r, "f"'] & B \arrow[u, "\subseteq"']
\end{tikzcd}$$

Pulling back the open cover of Y along f induces an open cover $f^{-1}(U)$, $f^{-1}(V)$ of the compact metric space $[0,1]^m$. The lemma of Lebesgue guarantees the existence of a natural number $N > 0$ such that the image of each m-cube

$$I^m_{k_1,\ldots,k_m} = [k_1/N, (k_1+1)/N] \times \ldots \times [k_m/N, (k_m+1)/N], \qquad 0 \leq k_i < N,$$

under f lies in U or in V. We now want to construct relative homotopies to modify f on those sub-cubes of the $I^m_{k_1,\ldots,k_m}$ which are not entirely mapped to V while we want to keep it unchanged on the remaining sub-cubes.

For this purpose, let us define a filtration on $X = I^m$,

$$\partial I^m \subseteq X^{(-1)} \subseteq X^{(0)} \subseteq \ldots \subseteq X^{(m)} = X = I^m,$$

as follows. Let J_{-1} be an index set for all l-sub-cubes, $0 \leq l \leq m$, of $I^m_{k_1,\ldots,k_m}$, $0 \leq k_i < N$, which are already completely mapped to V by f. Let us denote the l-sub-cube corresponding to such an index $\phi \in J_{-1}$ by I^l_ϕ. We then set

$$X^{(-1)} \quad = \quad \bigcup_{\phi \in J_{-1}} I^l_\phi,$$

and it follows from our assumption on f that $\partial I^m \subseteq X^{(-1)}$. We now have to take care of the remaining sub-cubes and this will be done by induction over the dimension of these sub-cubes. Thus, for each $0 \leq k \leq m$, let $J_k = \{\phi\}$ be an index set for all k-dimensional sub-cubes I^k_ϕ of the cubes $I^m_{k_1,\ldots,k_m}$ which satisfy $f(I^k_\phi) \nsubseteq V$. We then inductively set

$$X^{(k)} \quad = \quad X^{(k-1)} \cup \bigcup_{\phi \in J_k} I^k_\phi.$$

By definition, this gives us an exhaustive filtration of $X = I^m$ (which, in fact, defines a relative CW complex $(X, X^{(-1)})$).

We now want to inductively construct maps $h_k\colon X^{(k)} \to Y,\ k \geq -1$, such that:

(1) The map h_{-1} is obtained from f by restriction.
(2) The map h_k sends the cubes I^k_ϕ to $U \cap V$ for all $\phi \in J_k$ and $k \geq 0$.
(3) The map h_k extends h_{k-1}, i.e., we have $h_k|_{X^{(k-1)}} = h_{k-1}$ for all $k \geq 0$.

For h_0, note that $X^{(0)}$ is obtained from $X^{(-1)}$ by possibly adding some vertices which are mapped to U. For each such vertex, choose a path to a point in $U \cap V$. These target points together with h_{-1} then define the map h_0. For the inductive step, let us assume that h_{k-1} has already been constructed. For each $\phi \in J_k$, we have that $h_{k-1}(\partial I^k_\phi) \subseteq U \cap V$ (there are two different arguments for the two types of faces, one of them using that $N \in \mathbb{N}$ was chosen to be adapted to $f^{-1}(U)$ and $f^{-1}(V)$). Recall that by construction we have a homeomorphism $U \cap V \cong S^{n-1} \times (1/3, 2/3)$ so that we can our inductive assumption to find extensions as indicated in the following diagram:

$$\begin{tikzcd} \partial I^k_\phi \arrow[r, "h_{k-1}"] \arrow[d] & U \cap V \\ I^k_\phi \arrow[ur, dashed, "h_{k,\phi}"'] & \end{tikzcd}$$

It is easy to see that these maps $h_{k,\phi}$ and h_{k-1} can be assembled together in order to define a map $h_k\colon X^{(k)} \to Y$ with the desired properties. If we set $h = h_m\colon I^m = X^{(m)} \to Y$ then we have $h(I^m) \subseteq V$. Hence it suffices to show that $f \simeq h$ relative to ∂I^m.

We will in fact show that we can construct such a homotopy relative to $X^{(-1)}$. By construction, both maps f and h coincide on $X^{(-1)}$. Moreover, the restrictions of both maps to $X - X^{(-1)}$ can be considered as maps taking values in U. But, U is homeomorphic to an open n-disc, hence convex, so that the two restrictions are homotopic via linear homotopies. This homotopy together with the constant homotopy on $X^{(-1)}$ can be assembled together to give the desired homotopy $f \simeq h$ relative to $X^{(-1)}$ concluding the proof. $\square$