SECTION 10: CW APPROXIMATION AND WHITEHEAD'S THEOREM

In this section we will establish the two important theorems showing up in the title. The first of them, the theorem on the existence of CW approximations (Theorem 7), emphasizes the importance of CW complexes: up to weak equivalence *any* space can be replaced by a CW complex. Thus, if one is only interested in spaces up to this notion of equivalence, then it is enough to deal with CW complexes. The second theorem, the celebrated Whitehead theorem (Theorem 17), tells us that CW complexes are better behaved than arbitrary spaces in the following sense. The notions of weak homotopy equivalence and (actual) homotopy equivalence coincide if we only consider maps between CW complexes.

In both theorems the notion of a weak homotopy equivalence plays a key role so let us begin by introducing that concept.

Definition 1. A map of spaces $f\colon X \to Y$ is a **weak homotopy equivalence** if the induced maps

$$f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$$

are bijections for all dimensions k and *all* base points $x_0 \in X$.

Note that we insist that we have isomorphisms of homotopy groups for all points $x_0 \in X$. If one weakens this condition by considering a single base point only, then one obtains a different notion which we do not want to axiomatize here. The good notion is the one given above. Of course, the motivation for the terminology stems from the first point in the following exercise.

Exercise 2.

1. Let $f\colon X \to Y$ be a homotopy equivalence. Then f is a weak equivalence. (Note that the functoriality of the homotopy groups does not suffice to solve this part!)
2. Let $f\colon X \to Y$, $g\colon Y \to Z$ be maps of spaces, and let $h = gf\colon X \to Z$ be their composition. Show that if two of the maps f, g, and h are weak equivalences then so is the third one.
3. Two spaces X and Y are called **weakly equivalent** if there are finitely many weak equivalences

   $$X = X_0 \longrightarrow X_1 \longleftarrow X_2 \longrightarrow \cdots\cdots \longleftarrow X_{n-1} \longrightarrow X_n = Y$$

   pointing possibly in different directions which 'connect' X and Y. Check that this is an equivalence relation. The equivalence classes with respect to this equivalence relation are called **weak homotopy types**.
4. More generally, consider a relation $R \subseteq S \times S$. Define explicitly the equivalence relation $\sim_R$ on S generated by R, i.e., the smallest equivalence relation which contains R. Relate this to the previous part of the exercise (ignore set-theoretical issues for this comparison!).

There are the following classes of maps which allow us to measure how far a map is from being a weak equivalence.

Definition 3. Let $f\colon X \to Y$ be a map of spaces and let $n \geq 0$. Then f is an n**-equivalence** if for all $x_0 \in X$ the induced map

$$f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$$

is bijective for $k \leq n-1$ and surjective for $k = n$.

Thus, a map of spaces is a weak equivalence if and only if it is an n-equivalence for all $n \geq 0$. An interesting class of examples for this notion is given by the inclusions which are part of the skeleton filtration of a CW complex.

Lemma 4. *Let X be a CW complex and let $i_n \colon X^{(n)} \to X$ be the inclusion of the n-skeleton. Then i_n is an n-equivalence.*

Proof. This follows from a repeated application of the cellular approximation theorem. In order to obtain the surjectivity, consider a class $\alpha \in \pi_k(X, *)$ which can be represented by a cellular map $S^k \to X$. Thus, for all $k \leq n$, we can find a representative which factors over $i_n \colon X^{(n)} \to X$ showing that α lies in the image of $i_{n*} \colon \pi_k(X^{(n)}, *) \to \pi_k(X, *)$.

For the injectivity, consider two classes $\alpha,\ \beta \in \pi_k(X^{(n)}, *)$ for $k < n$, and represent them by cellular maps $f \colon S^k \to X^{(n)}$ and $g \colon S^k \to X^{(n)}$ respectively. By assumption, we can find a homotopy

$$H \colon S^k \times I \to X, \qquad H \colon f \simeq g.$$

Since I is compact, the space $S^k \times I$ is again a CW complex, and, by the explicit description of the CW structure, the subspace $S^k \times \partial I$ is a subcomplex. Now, the homotopy is a map which is already cellular on this subcomplex. Thus, an application of the cellular approximation theorem implies that we can find a cellular map $H' \colon S^k \times I \to X$ which restricts to f and g on the boundary components. Thus, this map factors over the inclusion i_n showing that $\alpha = \beta$ as intended. □

Exercise 5. Let (X, x_0) be a pointed, connected space, Y an arbitrary space, and $n \geq 0$. Then a map $f \colon X \to Y$ is an n-equivalence if and only if the induced map

$$f_* \colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$$

is bijective for $k \leq n - 1$ and surjective for $k = n$.

Let us now show that up to weak equivalence *every* topological space is a CW complex.

Definition 6. A **CW approximation** of a topological space X is a CW complex K together with a weak equivalence $f \colon K \to X$.

Theorem 7. ***(Existence of CW approximations)*** *Every space has a CW approximation.*

Proof. Let X be an arbitrary space. We can assume that the space X is path-connected by constructing a CW approximation for each path-component separately. It is easy to see that these CW approximations then assemble to one for the entire space.

We will now give an inductive construction of a CW approximation of X. More precisely, we will first construct n-equivalences

$$f_n \colon K_n \to X, \qquad n \geq 0,$$

for certain n-dimensional CW complexes K_n and then show that these maps can be assembled to a CW approximation $f \colon K \to X$.

In dimension $n = 0$ we let $K_0 = *$ be a single point and let $f_0 \colon K_0 \to X$ be the inclusion of an arbitrary point of X which obviously is a 0-equivalence. Let us assume inductively that we have already constructed an n-equivalence $f_n \colon K_n \to X$ with K_n an n-dimensional CW complex. We will construct the map f_{n+1} in two steps. First let us take care of the possibly non-trivial kernel

$$A_n = \ker\big(f_{n*} \colon \pi_n(K_n, *) \to \pi_n(X, *)\big).$$

Choose an arbitrary set of generators $(a_\sigma)_{\sigma \in J'_{n+1}}$ for the group A_n. Each generator can be represented by a map $\chi_\sigma \colon \partial e^{n+1} \to K_n$, and by definition of A_n we can choose homotopies $H_{n,\sigma}$

from $f_n \circ \chi_\sigma$ to a constant map. Now, construct the intermediate space K'_{n+1} by attaching $(n+1)$-cells to K_n as follows:

$$\begin{array}{ccc} J'_{n+1} \times \partial e^{n+1} & \xrightarrow{(\chi_\sigma)_\sigma} & K_n \\ \downarrow & & \downarrow i'_n \\ J'_{n+1} \times e^{n+1} & \longrightarrow & K'_{n+1} \end{array}$$

This way we obtain an $(n+1)$-dimensional CW complex K'_{n+1} such that $i'_n\colon K_n \to K'_{n+1}$ is the inclusion of the n-skeleton. Since K'_{n+1} is endowed with the quotient topology, it is easy to see that the homotopies $H_{n,\sigma}$ and the map f_n together induce a map $f'_{n+1}\colon K'_{n+1} \to X$ such that $f'_{n+1} \circ i'_n = f_n$. By Lemma 4 we know that i'_n is an n-equivalence, as is f_n by inductive assumption so that the same is also true for f'_{n+1}. Moreover, we can use the cellular approximation theorem to conclude that the induced map

$$f'_{n+1_*}\colon \pi_n(K'_{n+1},*) \to \pi_n(X,*)$$

is also injective. In fact, given an element α' in the kernel of that map, then there is a cellular map $S^n \to K'_{n+1}$ representing that class which hence factors as $S^n \to K_n \to K'_{n+1}$. We leave it to the reader to conclude from here that α' is trivial.

We next address the problem that the induced map might not be surjective in dimension $n+1$. Thus, let us consider the possibly non-trivial cokernel

$$B_{n+1} = \mathrm{coker}\big(f'_{n+1_*}\colon \pi_{n+1}(K'_{n+1},*) \to \pi_{n+1}(X,*)\big)$$

and let $(b_\sigma)_{\sigma \in J''_{n+1}}$ be a set of generators of B_{n+1}. Define K_{n+1} to be the wedge

$$K_{n+1} = K'_{n+1} \vee \bigvee_{\sigma \in J''_{n+1}} S^{n+1}.$$

Alternatively, this can also be described as an attachment of $(n+1)$-cells using constant attaching maps, i.e., we have a pushout diagram

$$\begin{array}{ccc} J''_{n+1} \times \partial e^{n+1} & \xrightarrow{(\kappa_*)} & K'_{n+1} \\ \downarrow & & \downarrow i''_n \\ J''_{n+1} \times e^{n+1} & \longrightarrow & K_{n+1}. \end{array}$$

In both descriptions (using the usual homeomorphism $e^{n+1}/\partial e^{n+1} \cong \partial e^{n+2}$ in the second one), the generators b_σ together with the map f'_{n+1} can be assembled to define a map $f_{n+1}\colon K_{n+1} \to X$ which satisfies $f_{n+1} \circ i''_n = f'_{n+1}\colon K'_{n+1} \to X$ and hence

$$f_{n+1} \circ i_n = f_n\colon K_n \to X$$

where $i_n = i''_n \circ i'_n\colon K_n \to K'_{n+1} \to K_{n+1}$. We leave it to the reader to check that f_{n+1} is an $(n+1)$-equivalence.

Thus, we have constructed n-dimensional CW complexes K_n together with n-equivalences f_n and inclusions $i_n\colon K_n \to K_{n+1}$ which are compatible with the n-equivalences. Let us denote by K the union $\bigcup_n K_n$ endowed with the weak topology. Then it is easy to see that K is a CW complex (with a single 0-cell and the set of n-cells given by $J_n = J'_n \sqcup J''_n$ for $n \geq 1$) such that its n-skeleton is given by $K^{(n)} = K_n$. The maps f_n induce a unique map $f\colon K \to X$ such that $f|_{K_n} = f_n$. The final claim is that f is a weak equivalence which we also leave to the reader (a further application of the cellular approximation theorem!). □

Exercise 8. Conclude the proof of Theorem 7 by establishing the following three steps (we use the notation of the proof):

(1) The map $f'_{n+1_*}\colon \pi_n(K'_{n+1}, *) \to \pi_n(X, *)$ is injective (and hence a bijection).
(2) The map $f_{n+1}\colon K_{n+1} \to X$ constructed in the induction step is an $(n+1)$-equivalence.
(3) The map $f\colon K \to X$ is a weak equivalence.

Perspective 9. We thus showed that every space is up to weak homotopy equivalence a CW complex. One might wonder if there is a functorial way of doing this. The first step would consist of the following problem. Let $X \to Y$ be a map of spaces and let $K \to X$ and $L \to Y$ be CW approximations of X and Y respectively. Can we then find a map $K \to L$ such that the following diagram commutes:

$$\begin{array}{ccc} K & \longrightarrow & X \\ {\scriptstyle \exists ?}\downarrow & & \downarrow \\ L & \longrightarrow & Y \end{array}$$

The first partially affirmative answer to this question (which lies only slightly beyond the scope of this course) is the following: we can always achieve this if we only insist that the square commutes up to homotopy, i.e., if we are asking for the existence of such a map such that both compositions are homotopic.

The second affirmative answer is even more positive. A construction of such a functorial CW approximation can be given using 'simplicial methods'. Given a space X one would consider all maps $\Delta^n \to X$ for the various $n \geq 0$ where Δ^n is the geometric n-simplex, i.e., the convex hull of the $n+1$ standard basis vectors of $\mathbb{R}^{n+1}$. For each n, one can single out a suitable subset $J_n \subseteq \hom_{\mathsf{Top}}(\Delta^n, X)$ such that these sets serve as index sets for a suitable CW complex. It can then be shown that these CW complexes are part of functorial CW approximation.

The 'simplicial methods' alluded to in the second affirmative answer are very powerful and show up in many areas of mathematics. In particular, the so-called simplicial sets –introduced in the 1950's– provide an interesting, purely combinatorial approach to homotopy theory whose importance in modern homotopy theory (and in other areas of mathematics) can hardly be overestimated.

The mapping cylinder construction allowed us in a previous lecture to show that every map can be factored into a cofibration followed by a strong deformation retraction. We would like to have a refinement of this result for the case of a cellular map between CW complexes. Recall that the mapping cylinder M_f of a map f is given by the following pushout construction:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_1}\downarrow & & \downarrow \\ X \times I & \longrightarrow & M_f \end{array}$$

Proposition 10. *The mapping cylinder M_f of a cellular map $f\colon X \to Y$ is again a CW complex which contains both X and Y as subcomplexes.*

Proof. We will not give a proof of this result but instead refer the reader to the book 'Cellular structures in topology' by Rudolf Fritsch and Renzo Piccinini. □

With this preparation we now obtain the following refinement of the factorization result.

Corollary 11. *Any cellular map can be factored as the inclusion of a CW subcomplex followed by a strong deformation retraction.*

A space is path-connected if any two points can be connected by a path, i.e., if π_0 applied to the space gives us a one-point set. Similarly, a space is called simply connected, if it is path-connected and has a trivial fundamental group (by the action of the fundamental groupoid it is not important which base point we consider). Let us generalize these definitions to higher dimensions.

Definition 12. A space X is n**-connected** if $\pi_k(X, x_0) \cong *$ for all $k \leq n$ and all $x_0 \in X$.

There is also a variant for pairs of spaces (X, A). Given an arbitrary point $a_0 \in A$ we gave a definition of $\pi_n(X, A, a_0) = \pi_n(X, A)$ in the case that $n \geq 1$. For $n \geq 2$ these are naturally groups which are abelian if $n \geq 3$. In fact, the definition of the underlying pointed set of $\pi_n(X, A)$ was as the set of homotopy classes of maps of triples

$$\pi_n(X, A) = [(I^n, \partial I^n, J^{n-1}), (X, A, a_0)]$$

where $J^{n-1} = I^{n-1} \times \{0\} \cup \partial I^{n-1} \times I \subseteq \partial I^n \subseteq I^n$. There are homeomorphisms $I^n/J^{n-1} \cong D^n$ and $\partial I^n/J^{n-1} \cong S^{n-1}$, and using these it is easy to show that we have natural bijections

$$\pi_n(X, A) \cong [(D^n, S^{n-1}, *), (X, A, a_0)].$$

Motivated by the long exact homotopy sequence of a pointed pair, let us say that $\pi_0(X, A) \cong *$ if the map $\pi_0(A, a_0) \to \pi_0(X, a_0)$ is surjective, i.e., if each path-component of X has a non-trivial intersection with A.

Definition 13. A pair of spaces (X, A) is n**-connected** if $\pi_k(X, A, a_0) \cong *$ for all $k \leq n$ and for all $a_0 \in A$.

We leave it as an exercise to establish the equivalence of the following statements.

Exercise 14. Let (X, A) be a pair of spaces and let $n \geq 0$. Then the following are equivalent:

(1) Every map $(D^n, S^{n-1}) \to (X, A)$ is homotopic relative to S^{n-1} to a map $D^n \to A$.
(2) Every map $(D^n, S^{n-1}) \to (X, A)$ is homotopic through such maps to a map $D^n \to A$.
(3) Every map $(D^n, S^{n-1}) \to (X, A)$ is homotopic through such maps to a constant map.
(4) We have $\pi_n(X, A, a_0) = \pi_n(X, A) \cong 0$ for all $a_0 \in A$.

This exercise is the basic building block for the following lemma which in turn is the key step towards the Whitehead theorem.

Lemma 15. *Let (X, A) be a relative CW complex and let (Y, B) be a pair of spaces with $B \neq \emptyset$ and such that $\pi_n(Y, B) = 0$ for all dimensions such that $X - A$ has n-cells. Then any map $f\colon (X, A) \to (Y, B)$ is homotopic relative A to a map with image in B.*

Proof. By assumption we have a filtration of X,

$$A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X,$$

such that the following two properties are satisfied:

(1) The space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells for $n \geq 0$.
(2) The space X is the union $\bigcup_{n \geq -1} X^{(n)}$ endowed with the weak topology and hence comes, in particular, with continuous inclusions $i_n\colon X^{(n)} \to X$.

The plan is to inductively construct intermediate maps $g_n\colon X \to Y$ such that $g_n(X^{(n)}) \subseteq B$ and homotopies $H_n\colon g_n \simeq g_{n-1}$ relative to $X^{(n-1)}$. We will then show how to conclude from here. By assumption the restriction $f_{-1}\colon A = X^{(-1)} \to Y$ already satisfies $f_{-1}(A) \subseteq B$ so that we set $g_{-1} = f_{-1}$.

Now let us assume inductively that maps g_k and homotopies H_k have already been constructed for $k < n$ and let $X - A$ have n-cells (otherwise the induction step is trivial) which we then index by a set J_n. For each $\sigma \in J_n$, let $\chi_\sigma\colon e^n \to X$ be an attaching map of the cell so that the square on the left is a pushout diagram:

$$\begin{array}{ccccc} J_n \times \partial e^n & \xrightarrow{(\chi_\sigma)} & X^{(n-1)} & \xrightarrow{g_{n-1}} & B \\ \downarrow & & \downarrow & & \downarrow \\ J_n \times e^n & \xrightarrow[(\chi_\sigma)]{} & X^{(n)} & \xrightarrow[g_{n-1}]{} & Y \end{array}$$

Now, each characteristic map induces an element $g_{n-1} \circ \chi_\sigma\colon (e^n, \partial e^n) \to (Y, B)$. Since by assumption $\pi_n(Y, B) \cong 0$ we can use Exercise 14 to obtain a homotopy $\widetilde{H}_{n,\sigma}\colon e^n \times I \to Y$ relative ∂e^n from $g_{n-1} \circ \chi_\sigma\colon e^n \to Y$ to a map $\widetilde{g}_{n,\sigma}$ which factors as $e^n \to B \to Y$. These homotopies together with the constant homotopy of g_{n-1} can be assembled to define a homotopy

$$\widetilde{H}_n\colon X^{(n)} \times I \to Y\colon g_{n-1} \simeq g_n \qquad \text{relative to } X^{(n-1)} \text{ with} \qquad g_n(X^{(n)}) \subseteq B.$$

Since the inclusion $i\colon X^{(n)} \to X$ is a cofibration, we can find a lift in the following diagram

$$\begin{array}{ccc} (g_{n-1}, \widetilde{H}_n)\colon X \times \{0\} \cup X^{(n)} \times I & \longrightarrow & Y \\ \downarrow & \overset{H_n}{\nearrow} & \\ X \times I & & \end{array}$$

Setting $g_n = H_n(-, 1)\colon X \to Y$ concludes the inductive step of the construction.

It remains to check that these infinitely many homotopies can be assembled into a single homotopy $H\colon X \times I \to Y$. In fact, as the homotopies H_n are relative to $X^{(n-1)}$, it follows that H_k is stationary on $X^{(n-1)}$ for $k \geq n$. Thus, we define H on $X^{(n-1)}$ by first running through H_0 at a double speed, then through H_1 at a fourfold speed, etc. We leave it to the reader to check that this way we obtain a continuous map $H\colon X \times I \to Y$. From the definition it is immediate that H is a homotopy relative to A and such that $H(X, 1) \subseteq B$ as intended. □

As an immediate consequence of this we collect the following convenient result.

Corollary 16. *Let (X, A) be a relative CW complex such that the inclusion $i\colon A \to X$ is a weak homotopy equivalence. Then i is the inclusion of a strong deformation retract.*

Proof. Apply the lemma to the identity morphism of (X, A). □

We can now use this lemma to establish the celebrated 'Whitehead's theorem'.

Theorem 17. ***(Whitehead's theorem)*** *Let $f\colon X \to Y$ be a weak equivalence between CW complexes X and Y. Then f is a homotopy equivalence. If f is the inclusion of a CW complex, then f is the inclusion of strong deformation retract.*

Proof. We leave it to the reader to reduce to the case of a path-connected CW complex. By the cellular approximation theorem we can assume that f is a cellular map. Moreover, the mapping cylinder construction allows us to assume that $f = i\colon X \to Y$ is the inclusion of a subcomplex. The long exact sequence of homotopy groups of the pair (Y, X) implies that all relative homotopy groups $\pi_n(Y, X)$ vanish. Thus the previous lemma applied to the identity $\mathrm{id}\colon (Y, X) \to (Y, X)$ implies that $\mathrm{id} \simeq i \circ r$ relative to X for some map $r\colon Y \to X$. Thus, this map r satisfies $r|_X = i$, i.e., $r \circ i = \mathrm{id}_X$. It follows that the map $i\colon X \to Y$ is the inclusion of a strong deformation retract, hence, in particular, a homotopy equivalence. □

Thus, from the knowledge about the behavior of a map at the level of homotopy groups we can actually *construct* a map in the converse direction. This indicates that the collection of invariants given by the homotopy groups at all points is very powerful.

Warning 18. Note however that the Whitehead theorem does **not** imply that two CW complexes X and Y are homotopy equivalent as soon as the corresponding homotopy groups $\pi_n(X)$ and $\pi_n(Y)$ are isomorphic for all $n \geq 0$. To put it differently, it does not suffice to have abstract isomorphisms of these groups. Instead, it is essential that these isomorphisms are –at least in one direction– induced by an actual map of spaces.

A close inspection of the proof of Theorem 7 shows that we also have the following refined version.

Corollary 19. *Let X be a n-connected space. Then there is a CW approximation $K \to X$ such that K has a trivial n-skeleton, i.e., such that $K^{(n)} = *$.*

A combination of this corollary with Whitehead's theorem gives the following nice fact.

Corollary 20. *A n-connected CW complex is homotopy equivalent to a CW complex with trivial n-skeleton.*

In the exercises, you will be asked to proof these two results. Using similar methods as above, one can also establish the following relative version of Whitehead's theorem.

Theorem 21. ***(relative version of Whitehead theorem)***
Let $f\colon (X, A) \to (Y, B)$ be a weak equivalence of relative CW complexes such that $f\colon A \to B$ is a homotopy equivalence. Then $f\colon (X, A) \to (Y, B)$ is a homotopy equivalence of pairs.