Algebraic K-theory
Exercise Set 1

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1. Prove that the group completion of a group is its abelienization.

2. A *monoid* is a semigroup equipped with a neutral element. An abelian monoid $(A, +, 0)$ is *free* on a subset $X \subseteq A$ if for every $a \in A$, there is exactly one function $f_a : X \to \mathbb{N}$ such that

   - $\#\{x \mid f_a(x) = 0\} < \infty$, and
   - $a = \sum_{x \in X} f_a(x) \cdot x$.

   Show that if $(A, +, 0)$ is free on X, then the group completion map $\gamma : A \to \widehat{A}$ is injective, and $\widehat{A}$ is a free abelian group, with basis $\{\gamma(x) \mid x \in X\}$.

3. Let $(S, *, e)$ be an abelian monoid, and define an equivalence relation $\sim$ on $S \times S$ by

   $$(s,t) \sim (s',t') \iff \exists u \in S \text{ s.t. } s * t' * u = t * s' * u.$$

   Let $s - t$ denote the equivalence class of (s,t) in $S \times S/\sim$.

   (a) Show that the binary operation

   $$(s-t) + (s'-t') := (s * s') - (t * t')$$

   on $S \times S/\sim$ is well defined and endows $S \times S/\sim$ with the structure of an abelian group.

   (b) Show that $S \times S/\sim$ is isomorphic to the usual group completion of $(S, *, e)$.

4. (A quick survey of projective modules) Let R be a ring, and let P be an R-module.

   (a) Show that the following conditions are equivalent.

      i. P is a direct summand of a free R-module.

ii. For every diagram

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow f & & \\ M & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}$$

of homomorphisms of R-modules in which the bottom row is exact (i.e., p is surjective), there is a homomorphism of R-modules $\widehat{f} : P \to M$ such that $p \circ \widehat{f} = f$.

iii. Every exact sequence of homomorphisms of R-modules

$$0 \to M \to N \to P \to 0$$

splits.

If any of the conditions above holds, we say that P is a *projective* R-module.

(b) Prove that if R is an integral domain, then every projective R-module is torsion-free.

(c) Prove that if R is a principal ideal domain, then every finitely generated, torsion-free R-module is projective.

(d) Prove $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.