Algebraic K-theory
Exercise Set 2

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1. Let R be a ring such that $0 \neq R$. Let $\mathrm{Mat}_\infty(R)$ denote the ring of functions

$$f : \mathbb{N} \times \mathbb{N} \to R$$

(which can be visualized as "$\mathbb{N} \times \mathbb{N}$-matrices") such that for every $j \in \mathbb{N}$

$$\#\{i \mid f(i,j) \neq 0\} < \infty.$$

Show that $K_0\big(\mathrm{Mat}_\infty(R)\big) = 0$.

2. Let R be a ring whose center contains a field $\mathbb{F}$. If $\dim_{\mathbb{F}} R < \infty$, prove that

$$\dim_{\mathbb{F}} : \mathcal{M}(R) \to (\mathbb{Z}, +, 0)$$

is a generalized rank.

Remark 0.1. Note that in this case the generalized rank does not send the base ring R to 1! An example of such a ring R is any polynomial ring with coefficients in a field $\mathbb{F}$.

3. Let p be a prime number, and let $\mathcal{C}$ be the full subcategory of ${}_{\mathbb{Z}}\mathbf{Mod}$, the objects of which are all finite, abelian p-groups. Show that $K_0(\mathcal{C}) \cong (\mathbb{Z}, +, 0)$.

4. Let R be a commutative PID.
   (a) Prove that if M is a free R-module of rank n, then every submodule of M is also free, of rank at most n.
   (b) Prove that $K_0(R) \cong (\mathbb{Z}, +, 0)$.

5. Let D be a division ring. Prove that $K_0(D) \cong (\mathbb{Z}, +, 0)$.