Algebraic K-theory
Exercise Set 3

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1. Prove the **Horseshoe Lemma**: Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. If
$$0 \to P'_n \to \cdots \to P'_0 \to L \to 0$$
and
$$0 \to P''_n \to \cdots \to P''_0 \to N \to 0$$
are projective resolutions, then there is a projective resolution
$$0 \to P'_n \oplus P''_n \to \cdots \to P'_0 \oplus P''_0 \to M \to 0.$$

2. Let R be a ring, and let $0 \to P_n \to \cdots \to P_0 \to 0$ be an exact sequence in $\mathcal{P}(R)$. Prove that
$$\sum_{i=0}^{n} (-1)^i r(P_i) = 0$$
for any generalized rank $r : \operatorname{Ob}\mathcal{P}(R) \to (A, +, 0)$.

3. Let R be a commutative PID.
   (a) Prove that if
   $$0 \to R^{\oplus n} \to R^{\oplus m} \to M \to 0 \text{ and } 0 \to R^{\oplus n'} \to R^{\oplus m'} \to M \to 0$$
   are both exact sequences of R-modules, then $m - n = m' - n'$. (Hint: Use Schanuel's Lemma and Exercise 4(a) of Exercise Set 2.)
   (b) Conclude that $G_0(R) \cong (\mathbb{Z}, +, 0)$.