Algebraic K-theory
Exercise Set 4

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1. Let $\mathbb{F}$ be a field, and let R be the ring of 2×2-matrices with coefficients in $\mathbb{F}$. Let
$$P = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{F} \right\}.$$
Prove that P is a projective R-module that is not stably free.

2. Let R be a ring such that $0 \neq R$. Let $\mathrm{Mat}_\infty(R)$ denote the ring of "$\mathbb{N} \times \mathbb{N}$-matrices" of Exercise 1 in Exercise Set 2. Let
$$P = \left\{ g : \mathbb{N} \to R \mid \#\{i \mid g(i) \neq 0\} < \infty \right\},$$
i.e., the elements of P can be seen as infinite column vectors with only finitely many nonzero coefficients. Show that $\mathrm{Mat}_\infty(R)$ acts on P on the left by "matrix multiplication" and that P is a a projective, stably free, but not free, R-module with respect to this action.

3. Let R be a ring, and let $P \in \mathcal{P}(R)$. Prove that P is stably free if and only if there is an exact sequence of R-modules
$$0 \to F_n \to \cdots \to F_0 \to P \to 0,$$
in which each F_i is a finitely generated, free R-module.