Algebraic K-theory
Exercise Set 7

15.04.2011

Remark 1. The point of the first exercise is to motivate our interest in local rings, at least within the context of algebraic K-theory. Indeed, part (f) of the exercise implies that if R is local, then $K_0(R) \cong K_0\big(\mathcal{F}(R)\big)$, so that its K-theory is, in some sense, particularly simple.

Since parts (a)-(c) of Exercise 1 are not directly related to the course material, you may simply assume they hold and use them to prove parts (d)-(f), if you choose.

1. (K_0 and quotients by radical ideals) Let R be a ring, and let $\mathcal{S}(R)$ be the collection of *simple* R-modules, i.e., $M \in \mathcal{S}(R)$ if and only if $M \neq 0$ and M has no nontrivial submodules. The *Jacobson radical* of R is the two-sided ideal

$$\operatorname{rad} R = \big\{ a \in R \mid aM = 0\ \forall M \in \mathcal{S}(R) \big\}.$$

An ideal I of R is *radical* if $I \subseteq \operatorname{rad} R$.

(a) Show that if R is commutative and $a \in R$ is nilpotent, then $a \in \operatorname{rad} R$.

(b) Prove **Nakayama's Lemma**: A left ideal I of R is radical if and only if $N = M$ whenever M is a finitely generated R-module of which N is a submodule such that $N + IM = M$.

(c) Use Nakayama's Lemma to prove that if I is a radical ideal of R and $P, Q \in \mathcal{P}(R)$, then

$$P/IP \cong Q/IQ \text{ as } R/I\text{-modules } \Longrightarrow P \cong Q \text{ as } R\text{-modules}.$$

(d) Prove that if I is a radical ideal of R, then the quotient homomorphism $q : R \to R/I$ induces an injective homomorphism

$$K_0(q) : K_0(R) \to K_0(R/I).$$

(e) Let I be a radical ideal of R. Prove that if all finitely generated, projective R/I-modules are (stably) free, then all finitely generated, projective R-modules are free.

(f) A ring R is *local* if $R/\operatorname{rad} R$ is a division ring. Conclude from (e) that
$$R \text{ local } \implies \mathcal{P}(R) = \mathcal{F}(R).$$

2. Let R be any ring, and let $(S, \cdot, 1)$ be a submonoid of $Z(R)$.

   (a) Show that $S^{-1}R = 0$ if and only if $0 \in S$. Conclude that if s is a nilpotent element of $Z(R)$, then $R[\frac{1}{s}] = 0$.

   (b) Show that the localization map $\iota_S : R \to S^{-1}R$ is injective if and only if S contains no zero divisors.

   (c) Show that ι_S is an isomorphism if and only if $S \subseteq R^*$.

   (d) Let spec(R) denote the set of prime ideals of R. Show that ι_S induces a bijection
$$\{\mathfrak{p} \in \operatorname{spec}(R) \mid \mathfrak{p} \cap S = \emptyset\} \xrightarrow{\cong} \operatorname{spec}(S^{-1}R).$$
   Conclude that $R_{\mathfrak{p}}$ is a local ring, for all $\mathfrak{p} \in \operatorname{spec}(R)$.

   (e) Apply Exercise 1(f) and a Grothendieck group computation from the course to show that $K_0(R_{\mathfrak{p}}) \cong \mathbb{Z}$.

3. Let R be a commutative ring, and let $\mathfrak{p} \in \operatorname{spec}(R)$. Let $rk_{\mathfrak{p}}$ denote the composite
$$\operatorname{Ob}\mathcal{P}(R) \xrightarrow{d_R} K_0(R) \xrightarrow{K_0(R_{\mathfrak{p}} \otimes_R -)} K_0(R_{\mathfrak{p}}) \cong \mathbb{Z},$$
   which is called the *local rank at* $\mathfrak{p}$.

   (a) If R is an integral domain, prove that $rk_{\mathfrak{p}} = rk_{\mathfrak{q}}$ for all $\mathfrak{p}, \mathfrak{q} \in \operatorname{spec}(R)$.

   (b) Find $\mathfrak{p}, \mathfrak{q} \in \operatorname{spec}(\mathbb{Q} \times \mathbb{Q})$ such that $rk_{\mathfrak{p}} \neq rk_{\mathfrak{q}}$.

4. Describe the K_0-localization sequence explictly when $S = \{1, e\}$, where e is a central idempotent of R.