Algebraic K-theory
Exercise Set 8

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1. Describe the K_0-localization sequence explictly when $S = \{1, e\}$, where e is a central idempotent of R.

2. Let $\Bbbk$ be a field, and let V be an infinite-dimensional $\Bbbk$-vector space. Let $R = \mathrm{End}_{\Bbbk}(V)$. Prove that $K_1 R$ is the trivial group.

   Hint: An Eilenberg swindle-type argument will do the trick. Observe first that $V \cong \bigoplus_{n \in \mathbb{N}} V$. It follows that any $A \in GL(R)$ gives rise to another element of $GL(R)$

$$\infty \cdot A = \begin{pmatrix} A & 0 & 0 & \cdots & & \\ 0 & A & 0 & \cdots & & \\ 0 & 0 & A & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix}.$$

   Show that A and $A \oplus \infty \cdot A$ are conjugate.

3. Let R be a Euclidean ring. Prove that $SK_1(R) = 0$ and thus that $K_1(R) \cong R^*$.

   Hint: Use the norm function on R to construct an inductive argument that any invertible matrix with coefficients in R can be row-reduced (essentially via the Gauss-Jordan algorithm) to an upper triangular matrix and then to a diagonal matrix and finally to a diagonal matrix with only one diagonal coefficient different from 1.

4. Apply exercise 3 to computing the following values of K_1.
   (a) $K_1(\mathbb{Z}) \cong \{1, -1\}$.
   (b) $K_1\big(\mathbb{Z}[i]\big) \cong \{1, i, -1, -i\}$.
   (c) $K_1\big(\mathbb{Z}[\frac{-1+i\sqrt{3}}{2}]\big) \cong \mu_6$, the multiplicative group of the 6^{th} roots of unity.
   (d) $K_1\big(\Bbbk[t]\big) \cong \Bbbk^*$, if $\Bbbk$ is a field.