Algebraic K-theory
Exercise Set 9

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1. Prove that $K_1\big(\mathcal{F}(-)\big)$ and $K_1\big(\mathcal{P}(-)\big)$ are natural, i.e., that they extend to functors
$$K_1\big(\mathcal{F}(-)\big), K_1\big(\mathcal{P}(-)\big) : \mathbf{Ring} \to \mathbf{Ab}.$$
Show that the isomorphisms $K_1R \cong K_1\big(\mathcal{F}(R)\big) \cong K_1\big(\mathcal{P}(R)\big)$ seen in class are natural.

2. Let $\mathbf{C}$ be a modest subcategory of ${}_R\mathbf{Mod}$, and let $M \in \mathbf{C}$, $\alpha \in \mathrm{Aut}(M)$. If
$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n$$
is a $\mathbf{C}$-filtration of M such that $\alpha(M_i) = M_i$ for all i, prove that
$$[M, \alpha] = \sum_{i=0}^{n-1} [M_i/M_{i+1}, \alpha_i]$$
in $K_1(\mathbf{C})$, where $\alpha_i \in \mathrm{Aut}(M_i/M_{i+1})$ is induced by α.

3. Let $\mathbf{C}$ be a modest subcategory of ${}_R\mathbf{Mod}$ such that if $f : M \to N$ is a morphism in $\mathbf{C}$, then $\ker f$ is an object in $\mathbf{C}$. Let
$$0 \to (M_n, \alpha_n) \to \cdots \to (M_0, \alpha_0) \to 0$$
be an exact sequence in $\mathbf{Aut}(\mathbf{C})$. Prove that
$$\sum_{i=0}^{n} (-1)^i [M_i, \alpha_i] = 0.$$

4. (A very brief introduction to the Whitehead group, which plays a crucial role in geometric topology.) Let G be a group, and let w denote the composite homomorphism
$$G \times \{1, -1\} \xrightarrow{j} GL_1\big(\mathbb{Z}[G]\big) \xrightarrow{s_1} K_1\big(\mathbb{Z}[G]\big),$$
where $j(g, \pm 1) = (\pm g)$. The *Whitehead group* of G is
$$Wh(G) := \mathrm{coker}\, w = K_1\big(\mathbb{Z}[G]\big) / < s_1(\pm g) \mid g \in G > .$$

(a) Prove that $Wh(\{e\}) \cong \{e\}$.

(b) Let C_2 denote the cyclic group of order 2. Prove that

$$Wh(C_2) \cong SK_1\big(D(\mathbb{Z}, (2))\big).$$

(c) (A bit tricky; see [Rosenberg, Thm. 2.4.3].) Prove that $SK_1\big(D(\mathbb{Z}, (2))\big)$ is in fact the trivial group, whence $Wh(C_2)$ is trivial.

(d) Let C_5 denote the cyclic group of order 5. Show that there is an element of infinite order in $Wh(C_5)$.

Hint: Let t denote the generator of C_5. Show that $a = 1 - t - t^{-1}$ is a unit in $\mathbb{Z}[G]$. Next show that the composite of the homomorphism $\alpha : \mathbb{Z}[G] \to \mathbb{C}$ specified by $\alpha(t) = e^{i2\pi/5}$ with the norm map $\mathbb{C} \to \mathbb{R}_{\geq 0}$ induces a homomorphism $Wh(C_5) \to (\mathbb{R}_{>0}, \cdot)$.

(e) Let G_{ab} denote the abelianization of G. Prove that there are group homomorphisms

$$G_{ab} \times \{1, -1\} \xrightarrow{j} K_1\big(\mathbb{Z}[G]\big) \xrightarrow{r} G_{ab} \times \{1, -1\}$$

such that $rj = \mathrm{Id}$. Conclude that

$$K_1\big(\mathbb{Z}[G]\big) \cong G_{ab} \times \{1, -1\} \times Wh(G).$$

Remark 0.1. In fact,

$$Wh(C_n) = \begin{cases} \{e\} & : n = 2, 3, 4, 6 \\ \mathbb{Z} & : n = 5. \end{cases}$$

More generally, if G is *finite* abelian, then the determinant induces an isomorphism

$$\det : K_1\big(\mathbb{Z}[G]\big) \to \mathbb{Z}[G]^*$$

and so

$$Wh(G) \cong \mathbb{Z}[G]^* / < \pm g \mid g \in G > .$$

On the other hand, Bass, Heller and Swan proved that if G is *free* abelian, then $Wh(G)$ is always trivial.