Homotopie et Homologie
Exercise Set 3

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Throughout these exercises, $\cong$ denotes homeomorphism of topological spaces or isomorphism of groups, and *space* means *topological space*.

1. Exercise 5 from Exercise Set 2.

2. Let T^2 be the 2-torus $S^1 \times S^1$. Show that every homomorphism

$$\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$$

can be *realized topologically* by a continous self-map of T^2, i.e., there exists $f : T^2 \to T^2$ such that $\pi_1 f = \varphi$.

3. (The Borsuk-Ulam Theorem) Use the isomorphism $\deg : \pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}$ to prove that if $f : S^2 \to \mathbb{R}^2$ is continous, then there exists $x \in S^2$ such that $f(x) = f(-x)$.

4. (This exercise and the next one use ideas introduced in Exercise 4 of Exercise set 13 from the second-year topology course.) Let X be any space. Consider the equivalence relation

$$R_{\text{path}} = \{(x, x') \in X \times X \mid \exists \lambda \in \text{Map}(I, X) \text{ s.t. } \lambda(0) = x, \lambda(1) = x'\}$$

on X. Henceforth, we write $x \sim x'$ if $(x, x') \in R$, and denote the equivalence class of x by $[x]$.

The *set of path-components of* X is the set

$$\pi_0 X := X/\sim$$

of equivalence classes of elements in X under the relation R_{path}.

Recall that if $f : X \to Y$ is continuous map, then

$$\pi_0 f : \pi_0 X \to \pi_0 Y : [x] \mapsto \big[f(x)\big]$$

is a well-defined function.

Prove that for all spaces X and Y, there is a bijection

$$\tau_{X,Y} : \pi_0 X \times \pi_0 Y \to \pi_0(X \times Y)$$

such that

$$\begin{array}{ccc} \pi_0 X \times \pi_0 Y & \xrightarrow{\tau_{X,Y}} & \pi_0(X \times Y) \\ \downarrow{\scriptstyle \pi_0 f \times \pi_0 g} & & \downarrow{\scriptstyle \pi_0(f \times g)} \\ \pi_0 X' \times \pi_0 Y' & \xrightarrow{\tau_{X',Y'}} & \pi_0(X' \times Y') \end{array}$$

commutes for all continuous maps $f : X \to X'$ and $g : Y \to Y'$.

5. Let X be a locally compact, Hausdorff space, and let Y be any topological space. Let $A \subseteq X$ and $B \subseteq Y$.

   (a) Show that

   $$\big[(X, A), (Y, B)\big] = \pi_0 \operatorname{Map}\big((X, A), (Y, B)\big).$$

   (b) Show that for any continuous map $f : X \to X'$, where X' is locally compact, Hausdorff and $f(A) \subseteq A' \subseteq X'$,

   $$\pi_0 f^\sharp = f^* : \big[(X', A'), (Y, B)\big] \to \big[(X, A), (Y, B)\big];$$

   (c) Show that if Y is locally compact, Hausdorff, then for any continuous map $g : Y \to Y'$ such that $g(B) \subseteq B' \subseteq Y'$,

   $$\pi_0 g_\sharp = g_* : \big[(X, A), (Y, B)\big] \to \big[(X, A), (Y', B')\big];$$

   (d) More generally, show that the composite

   $$\pi_0 \operatorname{Map}(X, Y) \times \pi_0 \operatorname{Map}(Y, Z) \xrightarrow{\tau} \pi_0\big(\operatorname{Map}(X, Y) \times \operatorname{Map}(Y, Z)\big) \xrightarrow{\pi_0 \gamma} \pi_0 \operatorname{Map}(X, Z)$$

   agrees with the definition of composition of homotopy classes of maps.

6. Let $\{X_j \mid j \in \mathscr{J}\}$ be a collection of spaces, and let $A_j \subseteq X_j$ for all $j \in \mathscr{J}$. Let $(X, A) = \big(\coprod_{j \in \mathscr{J}} X_j, \coprod_{j \in \mathscr{J}} A_j\big)$, where $\coprod$ denotes disjoint union.

   Show that there is a bijection

   $$\big[(X, A), (Y, B)\big] \to \prod_{j \in J} \big[(X_j, A_j), (Y, B)\big]$$

   for all spaces Y and subspaces $B \subseteq Y$.