Homotopie et Homologie
Exercise Set 6

27.10.2011

Throughout these exercises, *space* means *topological space.*

1. Let (X, x_0) be any based space. Check that the comultiplication and coinverse maps on $\Sigma(X, x_0)$ defined in class do indeed satisfy the axioms of a co-H-group.

2. Let $f : (X, x_0) \to (X', x_0')$ be a pointed, continuous map. Explain how to define $\Sigma f : \Sigma(X, x_0) \to \Sigma(X', x_0')$ and $\Omega f : \Omega(X, x_0) \to \Omega(X', x_0')$ so that Σf is a co-H-morphism and Ωf is an H-morphism.

3. It is easy to see that there are H-groups with strictly associative multiplication, strict units and strict inverses: any topological group (S^1, $GL(n, \mathbb{R})$, $O(n)$, $U(n)$,...) is an example of such. Show that, on the other hand, there are no nontrivial, strict co-H-groups.

4. Prove the *Characterization of co-H-groups*: A based space (X, x_0) admits the structure of a co-H-group if and only if the set $\big[(X, x_0), (Y, y_0)\big]_*$ admits a group structure natural in (Y, y_0).

5. Let (X, x_0, ψ) be a co-H-space, and let (Y, y_0) be any based space. Show that $\mathrm{Map}\big((X, x_0), (Y, y_0)\big)$ admits an H-space structure, which is natural in the sense that if $f : (Y, y_0) \to (Z, z_0)$ is a based, continuous map, then the induced map
$$f_* : \mathrm{Map}\big((X, x_0), (Y, y_0)\big) \to \mathrm{Map}\big((X, x_0), (Z, z_0)\big)$$
is an H-morphism.

6. For "any" pointed space (X, x_0), define pointed, continuous maps
$$\eta_{(X,x_0)} : (X, x_0) \to \Omega\Sigma(X, x_0)$$
and
$$\varepsilon_{(X,x_0)} : \Sigma\Omega(X, x_0) \to (X, x_0)$$
such that for all pointed, continuous maps $f : (X, x_0) \to (X', x_0')$,
$$\Omega\Sigma f \circ \eta_{(X,x_0)} = \eta_{(X',x_0')} \circ f : (X, x_0) \to \Omega\Sigma(X', x_0'),$$

and

$$f \circ \varepsilon_{(X,x_0)} = \varepsilon_{(X',x_0')} \circ \Sigma\Omega f : \Sigma\Omega(X,x_0) \to (X',x_0'),$$

while

$$\alpha : \mathrm{Map}\big(\Sigma(X,x_0),(X',x_0')\big) \to \mathrm{Map}\big((X,x_0),\Omega(X',x_0')\big) : g \mapsto \Omega g \circ \eta_{(X,x_0)}$$

and

$$\beta : \mathrm{Map}\big((X,x_0),\Omega(X',x_0')\big) \to \mathrm{Map}\big(\Sigma(X,x_0),(X',x_0')\big) : g \mapsto \varepsilon_{(X',x_0')} \circ \Sigma g$$

are mutually inverse homeomorphisms.