## Homotopie et Homologie Exercise Set 11

## 09.12.2010

Throughout these exercises, space means topological space, map means continuous map and I denotes [0, 1].

1. For any finite CW-complex X, let  $\zeta_n(X)$  denote the number of *n*-cells of X. The *Euler characteristic* of X is then

$$\chi(X) := \sum_{n \ge 0} (-1)^n \zeta_n(X)$$

Let A be a subcomplex of X, and let  $f : A \to Y$  be a cellular map, where Y is also finite.

- (a) Show that the pushout  $X \coprod_A Y$  of  $Y \xleftarrow{f} A \hookrightarrow X$  is a CW-complex. (Note: the finiteness assumptions are not necessary here.)
- (b) Find and prove a formula for  $\chi(X \coprod_A Y)$  in terms of  $\chi(A)$ ,  $\chi(X)$  and  $\chi(Y)$ .
- 2. The Lusternik-Schnirelmann category of a path-connected, pointed space  $(X, x_0)$ , denoted  $\operatorname{cat}(X)$ , is the least integer n such that X admits an open cover  $\{U_0, ..., U_n\}$  where each  $U_i$  is contractible in X, i.e., there exist a homotopies  $H_i: X \times I \to X$  such that  $H_i(-, 0) = Id_X$  and  $H_i(u, 1) = x_0$  for all  $u \in U_i$ .
  - (a) Prove that Lusternik-Schnirelmann category is a homotopy invariant.
  - (b) Prove that if X is a CW-complex of dimension n, then  $cat(X) \leq n$ .
- 3. We see in this exercise how to construct a canonical CW-approximation to a space X.

For any  $n \in \mathbb{N}$ , let

$$\Delta^{n} = \{(t_{0}, ..., t_{n}) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_{i} = 1, t_{i} \ge 0 \ \forall i\},\$$

and let

$$\partial^{i}: \Delta^{n-1} \to \Delta^{n}: (t_{0}, ..., t_{n-1}) \mapsto (t_{0}, ..., t_{i-1}, 0, t_{i}, ..., t_{n-1})$$

and

$$\sigma^{i}: \Delta^{n+1} \to \Delta^{n}: (t_{0}, ..., t_{n+1}) \mapsto (t_{0}, ..., t_{i} + t_{i+1}, ..., t_{n+1})$$

for all  $0 \leq i \leq n$ .

For any space X, let

$$S_n(X) = \{ \varphi : \Delta^n \to X \mid \sigma \text{ continuous } \}$$

seen as a discrete topological space, and

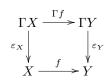
$$\Gamma X = \prod_{n \ge 0} S_n(X) \times \Delta^n / \sim,$$

where

$$(\varphi \circ \sigma^{i}, \mathbf{t}) \sim (\varphi, \sigma^{i}(\mathbf{t})) \text{ and } (\varphi \circ \partial^{i}, \mathbf{s}) \sim (\varphi, \partial^{i}(\mathbf{s}))$$

for all  $\varphi \in S_n(X)$ ,  $\mathbf{t} \in \Delta^{n+1}$ ,  $\mathbf{s} \in \Delta^{n-1}$ ,  $0 \le i \le n$  and  $n \ge 0$ . Endow  $\Gamma X$  with the obvious quotient topology.

- (a) Show that there is a CW-structure on  $\Gamma X$  such that the quotient map  $\coprod_{n\geq 0} S_n(X) \times \Delta^n \to \Gamma X$  is a cellular map.
- (b) Show that the evaluation maps  $ev: S_n(X) \times \Delta^n \to X: (\varphi, \mathbf{t}) \mapsto \varphi(\mathbf{t})$  together give rise to a continuous map  $\varepsilon_X: \Gamma X \to X$ .
- (c) Show that any map  $f:X\to Y$  gives rise to a map  $\Gamma f:\Gamma X\to \Gamma Y$  such that



commutes.

Remark 1. The map  $\varepsilon_X$  behaves very nicely: it satisfies a universal property (which we do not spell out here), and it is always a weak equivalence!