Homotopie et Homologie
Exercise Set 11

09.12.2010

Throughout these exercises, *space* means *topological space*, *map* means *continuous map* and I denotes $[0,1]$.

1. For any finite CW-complex X, let $\zeta_n(X)$ denote the number of n-cells of X. The *Euler characteristic* of X is then
$$\chi(X) := \sum_{n\geq 0}(-1)^n\zeta_n(X).$$
Let A be a subcomplex of X, and let $f : A \to Y$ be a cellular map, where Y is also finite.
   (a) Show that the pushout $X\coprod_A Y$ of $Y \xleftarrow{f} A \hookrightarrow X$ is a CW-complex. (Note: the finiteness assumptions are not necessary here.)
   (b) Find and prove a formula for $\chi(X\coprod_A Y)$ in terms of $\chi(A)$, $\chi(X)$ and $\chi(Y)$.

2. The *Lusternik-Schnirelmann category* of a path-connected, pointed space (X, x_0), denoted $\mathrm{cat}(X)$, is the least integer n such that X admits an open cover $\{U_0, ..., U_n\}$ where each U_i is contractible in X, i.e., there exist a homotopies $H_i : X \times I \to X$ such that $H_i(-,0) = Id_X$ and $H_i(u, 1) = x_0$ for all $u \in U_i$.
   (a) Prove that Lusternik-Schnirelmann category is a homotopy invariant.
   (b) Prove that if X is a CW-complex of dimension n, then $\mathrm{cat}(X) \leq n$.

3. We see in this exercise how to construct a canonical CW-approximation to a space X.
For any $n \in \mathbb{N}$, let
$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0\ \forall i\},$$
and let
$$\partial^i : \Delta^{n-1} \to \Delta^n : (t_0, ..., t_{n-1}) \mapsto (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1})$$

and

$$\sigma^i : \Delta^{n+1} \to \Delta^n : (t_0, ..., t_{n+1}) \mapsto (t_0, ..., t_i + t_{i+1}, ..., t_{n+1})$$

for all $0 \leq i \leq n$.

For any space X, let

$$S_n(X) = \{\varphi : \Delta^n \to X \mid \sigma \text{ continuous } \},$$

seen as a discrete topological space, and

$$\Gamma X = \coprod_{n \geq 0} S_n(X) \times \Delta^n / \sim,$$

where

$$(\varphi \circ \sigma^i, \mathbf{t}) \sim \big(\varphi, \sigma^i(\mathbf{t})\big) \text{ and } (\varphi \circ \partial^i, \mathbf{s}) \sim \big(\varphi, \partial^i(\mathbf{s})\big)$$

for all $\varphi \in S_n(X)$, $\mathbf{t} \in \Delta^{n+1}$, $\mathbf{s} \in \Delta^{n-1}$, $0 \leq i \leq n$ and $n \geq 0$. Endow ΓX with the obvious quotient topology.

(a) Show that there is a CW-structure on ΓX such that the quotient map $\coprod_{n \geq 0} S_n(X) \times \Delta^n \to \Gamma X$ is a cellular map.

(b) Show that the evaluation maps $ev : S_n(X) \times \Delta^n \to X : (\varphi, \mathbf{t}) \mapsto \varphi(\mathbf{t})$ together give rise to a continuous map $\varepsilon_X : \Gamma X \to X$.

(c) Show that any map $f : X \to Y$ gives rise to a map $\Gamma f : \Gamma X \to \Gamma Y$ such that

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Remark 1. The map ε_X behaves very nicely: it satisfies a universal property (which we do not spell out here), and it is always a weak equivalence!