Homotopie et Homologie
Exercise Set 12

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Throughout these exercises, *space* means *topological space*, and *map* means *continuous map*.

1. *Complex projective space* of dimenion n is

$$\mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\}/\sim$$

where $z \sim z'$ if and only if there exists $\lambda \in \mathbb{C}$ such that $z = \lambda z'$. Prove that $SP^n(S^2) \cong \mathbb{C}P^n$.

2. Let (X, x_0) and (Y, y_0) be pointed spacs. The canonical maps

$$r_X : X \vee Y \to X \text{ and } r_Y : X \vee Y \to Y$$

that send Y (respectively, X) to the basepoint, induce maps

$$SP\, r_X : SP(X \vee Y) \to SP\, X \text{ and } SP\, r_Y : SP(X \vee Y) \to SP\, Y$$

and therefore a map

$$\delta : SP(X \vee Y) \to SP\, X \times SP\, Y.$$

Show that δ is a weak homotopy equivalence and that therefore

$$\widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$$

for all n.

Definition 1. Let G be an abelian group, and let $n \geq 2$. An *Eilenberg-MacLane space of type* (G, n) is a pointed space (X, x_0) such that $\pi_n(X, x_0) \cong G$ and $\pi_k(X, x_0)) = 0$ if $k \neq n$. A *Moore space of type* (G, n) is a pointed space (X, x_0) such that $\widetilde{H}_n(X, x_0) \cong G$ and $\widetilde{H}_k(X, x_0)) = 0$ if $k \neq n$.

Conventions 2. An Eilenberg-MacLane space of type (G, n) is usually denoted $K(G, n)$, while a Moore space of type (G, n) is usually denoted $M(G, n)$.

Remark 3. Recall that for all $n \geq 1$, S^n is a Moore space of type $(\mathbb{Z}, n)$, i.e., that $SP(S^n)$ is an Eilenberg-MacLane space of type $(\mathbb{Z}, n)$.

3. Let $m \in \mathbb{N}$, and define $\lambda_m : S^1 \to S^1$ by $\lambda_m(e^{i\theta}) = e^{im\theta}$.
   (a) Prove that $\pi_1\lambda_m : \pi_1 S^1 \to \pi_1 S^1$ is given by multiplication by m.
   (b) Prove that C_{λ_m} is a a Moore space of type $(\mathbb{Z}/m, 1)$, i.e., that $SP(C_{\lambda_m})$ is an Eilenberg-MacLane space of type $(\mathbb{Z}/m, 1)$.
   Hint 4. Use the quasi-fibration obtained by applying SP to the sequence $S^1 \xrightarrow{\lambda_m} S^1 \to C_{\lambda_m}$.
   (c) Let $n > 1$. Prove that $\Sigma^{n-1}C_{\lambda_m}$ is a Moore space of type $(\mathbb{Z}/m, n)$, i.e., that $SP(\Sigma^{n-1}C_{\lambda_m})$ is an Eilenberg-MacLane space of type $(\mathbb{Z}/m, n)$.
   (d) Let G be any finitely generated abelian group, and let $n \geq 1$. Explain how to construct a Moore space of type (G, n) and an Eilenberg-MacLane space of the same type.

4. Let $f : (X, x_0) \to (Y, y_0)$ be a map of CW-complexes. Prove that there is a long exact sequence

$$\cdots \xrightarrow{\widetilde{H}_{n+1}i} \widetilde{H}_{n+1}C_f \xrightarrow{\partial_{n+1}} \widetilde{H}_nX \xrightarrow{\widetilde{H}_nf} \widetilde{H}_nY \xrightarrow{\widetilde{H}_ni} \widetilde{H}_nC_f \xrightarrow{\partial_n} \cdots \xrightarrow{\widetilde{H}_1i} \widetilde{H}_1C_f \xrightarrow{\partial_1} \widetilde{H}_0X \xrightarrow{\widetilde{H}_0f} \widetilde{H}_0Y \xrightarrow{\widetilde{H}_0i} \widetilde{H}_0C_f.$$

This the *long exact sequence in (reduced) homology* associated to a map.

5. Prove that for all CW-pairs (X, A), the sequence

$$\cdots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial_{n+1}} H_nA \longrightarrow H_nX \longrightarrow H_n(X,A) \xrightarrow{\partial_n} \cdots \longrightarrow H_1(X,A) \xrightarrow{\partial_1} \widetilde{H}_0A \longrightarrow \widetilde{H}_0X \longrightarrow H_0(X,A)$$

   is exact.
   Hint 5. To prove exactness at H_nX, $H_n(X, A)$ and H_nA, apply the sequence of Exercise 4 to the inclusions $A_+ \hookrightarrow X_+$ and $X \hookrightarrow X_+ \coprod_{A_+} CA_+$ and to the quotient map $X_+ \coprod_{A_+} CA_+ \to \Sigma A_+$, respectively.

6. Prove that excision for excisive triads is equivalent to the following property.

   Let (X, A) be a pair of spaces. For all $U \subseteq A$ satisfying $\overline{U} \subseteq \mathring{A}$, the inclusion $(X \smallsetminus U, A \smallsetminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$H_n(X, A) \cong H_n(X \smallsetminus U, A \smallsetminus U) \quad \forall n \geq 0.$$

   In other words, one can *excise* a nice enough subset without changing the homology of the pair.