Homotopie et Homologie
Exercise Set 13

23.12.2010

Throughout these exercises, *space* means *topological space*, and *map* means *continuous map*.

1. Let G be an abelian group, and let $M(G,n)$ be a Moore space of type (G,n) (cf. Exercise 1, Exercise set 13). The n^{th}*-reduced homology group with coefficients in* G of a pointed space (X,x_0) is
$$\widetilde{H}_n(X;G) := \pi_n\Big(SP\big(X \wedge M(G,1)\big)\Big).$$
Prove that $\{\widetilde{H}_n(-;G) \mid n \geq 0\}$ satisfies the Eilenberg-Steenrod axioms for a reduced homology theory, with corresponding Dimension Axiom
$$\widetilde{H}_n(S^0;G) = \begin{cases} G & : n = 0 \\ 0 & : n \neq 0. \end{cases}$$

2. For any pointed space (X,x_0), let $h_X : \pi_* X \to \widetilde{H}_* X$ be the Hurewicz homomorphism. Let $\iota_n = h_{S^n}\big([Id_{S^n}]\big)$, for all $n \geq 0$. Prove that for all $[\alpha] \in \pi_n X$,
$$h_X\big([\alpha]\big) = H_n(\alpha)(\iota_n).$$
Remark 1. When another, less "homotopical" approach has been taken to defining homology groups, the formula above is often given as the *definition* of the general Hurewicz homomorphism, once it has been specified on spheres.

3. Prove the following result, which can be seen as analogous to the Seifert-van Kampen Theorem.

Theorem 2 (Mayer-Vietoris)**.** *If* $(X;A,B)$ *is an excisive triad, then the sequence*
$$\cdots \xrightarrow{\Delta_{n+1}} H_n(A \cap B) \xrightarrow{\psi_n} H_n(A) \oplus H_n(B) \xrightarrow{\varphi} H_n X \xrightarrow{\Delta_n} \cdots$$
is exact, where

- $\psi_n(c) = \big(H_n(i)(c), H_n(j)(c)\big)$, *where* $i : A \cap B \hookrightarrow A$ *and* $j : A \cap B \hookrightarrow B$ *are the inclusions;*
- $\varphi_n(a, b) = H_n(k)(a) - H_n(\ell)(b)$, *where* $k : A \hookrightarrow X$ *and* $\ell : B \hookrightarrow X$ *are the inclusions; and*
- Δ *is the composite*

$$H_n X \to H_n(X, B) \cong H_n(A, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B),$$

where ∂ *is the connecting homomorphism of the Exactness Axiom.*

4. Recall the definition of complex projective space $\mathbb{C}P^n$ from Exercise 1, Exercise set 12.
   (a) Show that $\mathbb{C}P^n$ admits a CW-decomposition with exactly one $2k$-cell for all $0 \leq k \leq n$ and no odd-dimensional cells.
   (b) Use the Mayer-Vietoris Theorem to compute $H_* \mathbb{C}P^n$.

Joyeuses fêtes de fin d'année!!