Homotopie et Homologie
Exercise Set 2

30.09.2010

Throughout these exercises, $\cong$ denotes homeomorphism of topological spaces or isomorphism of groups, and *space* means *topological space*.

1. Exercises 6, 7, and 8 from Exercise Set 1!

2. Let (X, x_0) and (Y, y_0) be pointed spaces. Show that there is an isomorphism
$$\pi_1\big(X \times Y, (x_0, y_0)\big) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

3. Let T^2 be the 2-torus $S^1 \times S^1$. Show that every homomorphism
$$\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$$
can be *realized topologically* by a continous self-map of T^2, i.e., there exists $f : T^2 \to T^2$ such that $\pi_1 f = \varphi$.

4. Let $f : (X, x_0) \to (Y, y_0)$ be a based, continuous map. If f is surjective (respectively, injective), is the induced homomorphism $\pi_1 f : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ necessarily surjective (respectivement, injective)?

5. (The Brouwer Fixed Point Theorem) Let $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$. Use the fact that $\pi_1(S^1, 1) \cong \mathbb{Z}$ to prove that every continuous map $f : D^2 \to D^2$ has a fixed point, i.e., there exists $z \in D^2$ such that $f(z) = z$. (Hint: prove first that the inclusion $S^1 \hookrightarrow D^2$ does not admit a retraction, i.e., there is no continous map $r : D^2 \to S^1$ such that $r \circ i = Id_{S^1}$.)

6. (The Borsuk-Ulam Theorem) Use the isomorphism $\deg : \pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}$ to prove that if $f : S^2 \to \mathbb{R}^2$ is continous, then there exists $x \in S^2$ such that $f(x) = f(-x)$.