Homotopie et Homologie
Exercise Set 3

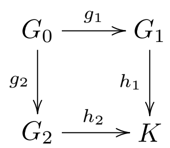
07.10.2010

Throughout these exercises, I denotes the unit interval $[0, 1]$, $\cong$ denotes homeomorphism of topological spaces or isomorphism of groups, and *space* means *topological space*.

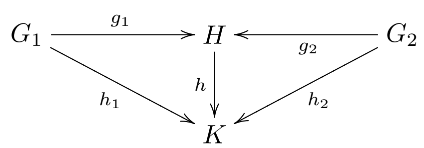
1. For any positive integer n, explain how to construct a path-connected space X_n from a circle and a disk such that $\pi_1(X_n, x_0) \cong \mathbb{Z}/n\mathbb{Z}$. Show that X_2 is homeomorphic to $\mathbb{R}P^2$, the *real projective plane*, which is usually defined as the quotient of D^2 by the relation that identifies antipodal points on its boundary.

2. Prove that $G * \{e\} \cong G$ for every group G.

3. Let $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ be group homomorphisms. The *pushout* (or *amalgamated sum*) of f_1 and f_2 consists of a group H and two homomorphisms $g_1 : G_1 \to H$ and $g_2 : G_2 \to H$ such that $g_1 f_1 = g_2 f_2$ and such that for any pair of homomorphisms $h_1 : G_1 \to K$ and $h_2 : G_2 \to K$ such that



commutes, there is a unique homomorphism $h : H \to K$ such that

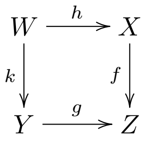


commutes. Prove that pushouts of pairs of group homomorphisms exist and are unique up to isomorphism.

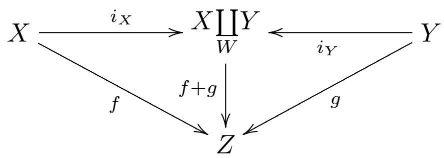
4. Let $h : W \to X$ and $k : W \to Y$ be continuous maps. The *pushout* of h and k is the quotient space

$$X \coprod_W Y = X \coprod Y / \sim,$$

where $\sim$ is the equivalence relation generated by $h(w) \sim k(w)$ for all $w \in W$. Show that for any pair of continuous maps $f : X \to Z$ and $g : Y \to Z$ such that



commutes, there is a unique continuous map $f + g : X \coprod_W Y \to Z$ such that

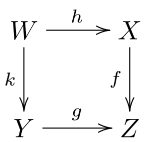


commutes. Formulate and prove a pointed version of this result.

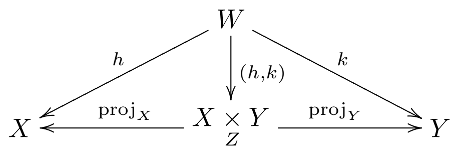
5. Let $f : X \to Z$ and $g : Y \to Z$ be continuous maps. The *pullback* of f and g is the space

$$X \underset{Z}{\times} Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

topologized as a subspace of $X \times Y$. Show that for any pair of continuous maps $h : W \to X$ and $k : W \to Y$ such that



commutes, there is a unique continuous map $(h, k) : W \to X \underset{Z}{\times} Y$ such that



commutes. Formulate and prove a pointed version of this result.