Homotopie et Homologie
Exercise Set 4

14.10.2010

Throughout these exercises, *space* means *topological space.*

1. Let (X, x_0, μ) be an H-space. Show that if $f : (X, x_0) \to (X', x_0')$ is a based homotopy equivalence, then X' admits an H-space structure with respect to which f is an H-morphism. Show that, moreover, if X is actually an H-group, then so is X'. In other words, "to be an H-space (respectively, H-group)" is a homotopy invariant notion.

2. Let (X, x_0, μ) be an H-space, and let (Y, y_0) be any based space. Show that $\mathrm{Map}\big((Y, y_0), (X, x_0)\big)$ admits an H-space structure, which is natural in the sense that if $f : (Y, y_0) \to (Z, z_0)$ is a based, continuous map, then the induced map
$$f^* : \mathrm{Map}\big((Z, z_0), (X, x_0)\big) \to \mathrm{Map}\big((Y, y_0), (X, x_0)\big)$$
is an H-morphism.

3. Let (X, x_0) be any based space. Check that the multiplication and inverse maps on $\Omega(X, x_0)$ defined in class do indeed satisfy the axioms of an H-group. Show furthermore that $\Omega^2(X, x_0)$ is a homotopy commutative H-group.

4. Prove that any H-morphism of H-groups preserves homotopy inversion up to homotopy, i.e., if (X, x_0, μ, σ) and $(X', x_0', \mu', \sigma')$ are H-groups and $f : (X, x_0, \mu) \to (X', x_0', \mu')$ is an H-morphism, then $f\sigma \simeq_* \sigma' f$.

5. Let (X, x_0, μ) and (X', x_0', μ') be H-spaces. Prove that the cartesian product $X \times X'$ admits an H-multiplication μ''
   (a) with respect to which the projection maps from $X \times X' \to X$ and $X \times X' \to X'$ are H-morphisms, and
   (b) such that the unique continuous map $(f, f') : Y \to X \times X'$ induced by a pair of H-maps $f : (Y, y_0, \nu) \to (X, x_0, \mu)$ and $f' : (Y, y_0, \nu) \to (X', x_0', \mu')$ is also an H-map.