Homotopie et Homologie
Exercise Set 5

21.10.2010

Throughout these exercises, *space* means *topological space.*

1. Let (X, x_0, ψ) be a co-H-space. Show that if $f : (X, x_0) \to (X', x'_0)$ is a based homotopy equivalence, then X' admits a co-H-space structure with respect to which f is a co-H-morphism. Show that, moreover, if X is actually a co-H-group, then so is X'. In other words, "to be a co-H-space (respectively, co-H-group)" is a homotopy invariant notion.

2. Let (X, x_0, ψ) be a co-H-space, and let (Y, y_0) be any based space. Show that $\mathrm{Map}\big((X, x_0), (Y, y_0)\big)$ admits an H-space structure, which is natural in the sense that if $f : (Y, y_0) \to (Z, z_0)$ is a based, continuous map, then the induced map

$$f_* : \mathrm{Map}\big((X, x_0), (Y, y_0)\big) \to \mathrm{Map}\big((X, x_0), (Z, z_0)\big)$$

   is an H-morphism.

3. Let (X, x_0) be any based space. Check that the comultiplication and coinverse maps on $\Sigma(X, x_0)$ defined in class do indeed satisfy the axioms of a co-H-group.

4. It is easy to see that there are H-groups with strictly associative multiplication, strict units and strict inverses: any topological group (S^1, $GL(n, \mathbb{R})$, $O(n)$, $U(n)$,...) is an example of such. Show that, on the other hand, there are no nontrivial, strict co-H-spaces.

5. For any based, continuous map $f : (X, x_0) \to (Y, y_0)$, show that $C_f/Y \cong \Sigma(X, x_0)$ (cf. Exercise Set 1, # 8).

6. Let $f : (X, x_0) \to (X', x'_0)$ be a pointed, continuous map. Explain how to define $\Sigma f : \Sigma(X, x_0) \to \Sigma(X', x'_0)$ and $\Omega f : \Omega(X, x_0) \to \Omega(X', x'_0)$ so that Σf is a co-H-morphism and Ωf is an H-morphism.

7. For "any" pointed space (X, x_0), define pointed, continuous maps

$$\eta_{(X,x_0)} : (X, x_0) \to \Omega\Sigma(X, x_0)$$

and

$$\varepsilon_{(X,x_0)} : \Sigma\Omega(X,x_0) \to (X,x_0)$$

such that for all pointed, continuous maps $f:(X,x_0)\to(X',x_0')$,

$$\Omega\Sigma f\circ\eta_{(X,x_0)} = \eta_{(X',x_0')}\circ f : (X,x_0)\to\Omega\Sigma(X',x_0'),$$

and

$$f\circ\varepsilon_{(X,x_0)} = \varepsilon_{(X',x_0')}\circ\Sigma\Omega f : \Sigma\Omega(X,x_0)\to(X',x_0'),$$

while

$$\alpha : \mathrm{Map}\big(\Sigma(X,x_0),(X',x_0')\big)\to\mathrm{Map}\big((X,x_0),\Omega(X',x_0')\big) : g\mapsto\Omega g\circ\eta_{(X,x_0)}$$

and

$$\beta : \mathrm{Map}\big((X,x_0),\Omega(X',x_0')\big)\to\mathrm{Map}\big(\Sigma(X,x_0),(X',x_0')\big) : g\mapsto\varepsilon_{(X',x_0')}\circ\Sigma g$$

are mutually inverse homeomorphisms.