Homotopie et Homologie
Exercise Set 8

11.11.2010

Throughout these exercises, *space* means *topological space*, *map* means *continuous map*, and I denotes $[0,1]$.

1. Prove that if an inclusion $j : A \hookrightarrow X$ is a cofibration, then so is
$$j \times Id_Z : A \times Z \to X \times Z$$
for all spaces Z.

2. Prove that if $i : A \hookrightarrow B$ and $j : B \hookrightarrow X$ are cofibrations, then so is $j \circ i : A \hookrightarrow X$. Prove more generally that if
$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \cdots$$
are inclusions that are cofibrations, then the inclusion $A_0 \hookrightarrow \bigcup_{n\geq 0} A_n$ is a cofibration, if the topology on $\bigcup_{n\geq 0} A_n$ satisfies:
$$C \subseteq \bigcup_{n\geq 0} A_n \text{ closed } \Leftrightarrow C \cap A_n \text{ closed in } A_n \forall n.$$

3. Prove that an inclusion $j : A \hookrightarrow X$ is a cofibration if and only if the map
$$\mathrm{Map}(X \times I, Y) \to \mathrm{Map}(X, Y) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(A \times I, Y)$$
induced by
$$j_0^* : \mathrm{Map}(X \times I, Y) \to \mathrm{Map}(X, Y) : H \mapsto H \circ j_0$$
and
$$(j \times Id_I)^* : \mathrm{Map}(X \times I, Y) \to \mathrm{Map}(A \times I, Y) : H \mapsto H \circ (j \times Id_I)$$
is surjective.

4. Let

$$\begin{tikzcd} B \arrow[d, "\beta"'] & A \arrow[l, "f"'] \arrow[r, "j"] \arrow[d, "\alpha"'] & X \arrow[d, "\gamma"'] \\ B' & A' \arrow[l, "f'"'] \arrow[r, "j'"] & X' \end{tikzcd}$$

be a commutative diagram of continuous maps.

(a) Prove the *Gluing Lemma*: If j and j' are cofibrations, and α, β and γ are homotopy equivalences, then the induced map on pushouts

$$\gamma + \beta : X \coprod_A B \to X' \coprod_{A'} B'$$

is a homotopy equivalence as well.

(b) Find a counter-example when at least one of j and j' is not a cofibration.

5. Consider a pushout diagram of spaces

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "j'"] \\ X \arrow[r, "f'"] & X \coprod_A B, \end{tikzcd}$$

where j is a cofibration.

(a) Show that if j is a homotopy equivalence, then so is j'.

(b) Show that if f is a homotopy equivalence, then so is f'.