Homotopie et Homologie
Exercise Set 9

18.11.2010

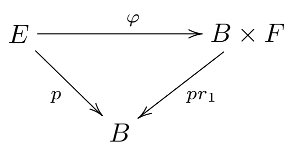
Throughout these exercises, *space* means *topological space*, *map* means *continuous map*, and I denotes $[0,1]$.

1. Prove that a composite of fibrations is a fibration.

2. Let $p : E \to B$ be a Hurewicz fibration. Prove that if B is path-connected, then for all $b_0, b_1 \in B$,
$$p^{-1}(b_0) \sim p^{-1}(b_1).$$
Hint 1. Use the path lifting map $\Gamma : E \times_B \mathrm{Map}(I, B) \to \mathrm{Map}(I, E)$ associated to p.

3. Let $p : E \to B$ be a Hurewicz fibration, and let $b_0 \in B$. Let $F = p^{-1}(b_0)$.

   (a) Prove that if $B \simeq \{b_0\}$, then there is a homotopy equivalence $\varphi : E \to B \times F$ such that

   commutes.

   Hint 2. Use the path lifting map $\Gamma : E \times_B \mathrm{Map}(I, B) \to \mathrm{Map}(I, E)$ associated to p and the function $\alpha : \mathrm{Map}(B \times I, B) \to \mathrm{Map}\big(B, \mathrm{Map}(I, B)\big)$ from the very first lecture.

   Remark 3. For the remainder of the exercise, we no longer assume that B is contractible.

   (b) Prove that the homotopy fiber of p (cf. Definition 2, Exercise set 6) is homotopy equivalent to F.

   (c) Prove that there is an exact sequence
$$\cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots \to \pi_1 B \to \pi_0 F \to \pi_0 E \to \pi_0 B$$

(where homotopy groups of B are calculated with respect to b_0, while those of E and F are calculated with respect to some $e_0 \in F$) and describe the homomorphisms in the sequence (cf. Exercise 2, Exercise set 7). This is the *long exact sequence in homotopy* of the fibration p.

Remark 4. In the textbook, the authors prove the existence of such long exact sequences for all Serre fibrations as well (Corollary 4.3.34).

4. Let A be a closed subspace of a locally compact, Hausdorff space X. Prove that if the inclusion $j : A \to X$ is a cofibration, then

$$j^* : \mathrm{Map}(X, Y) \to \mathrm{Map}(A, Y)$$

verifies the $\mathcal{C}$-homotopy lifting property, where $\mathcal{C}$ is the class of locally compact spaces.

5. Let

$$\begin{array}{ccccc} X & \xrightarrow{f} & B & \xleftarrow{p} & E \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\ X' & \xrightarrow{f'} & B' & \xleftarrow{p'} & E' \end{array}$$

be a commutative diagram of continuous maps.

(a) Prove the *Cogluing Lemma*: If p and p' are fibrations, and α, β and γ are homotopy equivalences, then the induced map on pullbacks

$$(\gamma, \alpha) : E \times_B X \to E' \times_{B'} X'$$

is a homotopy equivalence as well.

(b) Find a counter-example when at least one of p and p' is not a fibration.

6. Consider a pullback diagram of spaces

$$\begin{array}{ccc} E \times_B X & \xrightarrow{f'} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B. \end{array}$$

(a) Show that if p is a homotopy equivalence, then so is p'.

(b) Show that if f is a homotopy equivalence, then so is f'.