

## LECTURE 11: CELLULAR HOMOLOGY

In this lecture we continue the study of homological properties of CW complexes, culminating in the definition of *cellular homology* for such complexes. In the next lecture, we show that this alternative homology theory is naturally isomorphic to singular homology and that it is useful in explicit calculations.

We begin by recalling some basics about (homological) orientations. Recall that  $H_n(S^n) \cong \mathbb{Z}$ . An *orientation* of  $S^n$  is a choice of generator in  $H_n(S^n)$ ; so there are two orientations. The boundary  $\partial\Delta^n$  of the  $n$ -simplex is a model of  $S^{n-1}$ , and has a canonical orientation given by the order of its vertices

$$v_0, \dots, v_n$$

where  $v_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$  with 1 in the  $i$ -th place,  $i = 0, \dots, n$ . More precisely, the  $(n-1)$ -cycle  $\sum (-1)^i \partial_i$  is a generator, where  $\partial_i: \Delta^{n-1} \rightarrow \Delta^n$  is the face opposite to the  $i$ -th vertex,

$$\partial_i(x_0, \dots, x_{n-1}) = (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}).$$

An *orientation* of the  $n$ -cell  $e^n$  is a generator of  $H_n(e^n, \partial e^n) \cong \mathbb{Z}$  by the long exact sequence of the pair and the contractibility of  $e^n$ . Each homeomorphism  $\alpha: \Delta^n \rightarrow e^n$  determines an orientation, since the map  $\alpha$  itself is a cycle and represents an element of  $H_n(e^n, \partial e^n)$ .

An **oriented  $n$ -cell** in a CW complex  $X$  is a pair  $(e, \theta)$  consisting of an  $n$ -cell  $e$  in  $X$  and an orientation  $\theta$  of  $e$ . We write  $C_n^{\text{or}}(X)$  for the free abelian group generated by the oriented  $n$ -cells of  $X$ . Let  $C_n^{\text{cell}}(X)$  be the quotient of  $C_n^{\text{or}}(X)$  obtained by identifying  $(e, \theta)$  and  $-(e, \theta')$  if  $\theta$  and  $\theta'$  are the two possible orientations of  $e$ . So  $C_n^{\text{cell}}(X)$  is isomorphic to the free abelian group on the set of  $n$ -cells (but the isomorphism would require a choice of orientations).

In the final lecture we will prove the following theorem.

**Theorem 1.** *Let  $X$  be a CW complex. The abelian groups  $C_n^{\text{cell}}(X)$  can be turned into a chain complex, the homology of which is isomorphic to the singular homology  $H_n(X)$  of  $X$ .*

Of course a complete statement of the theorem, and its proof, requires an explicit description of the (cellular) boundary operator

$$\partial: C_n^{\text{cell}}(X) \rightarrow C_{n-1}^{\text{cell}}(X).$$

This description will be given in the next lecture and is based on the homological degree of maps  $S^n \rightarrow S^n$ . But even as it stands, it is already clear that the theorem is useful in calculations. For example, for the complex projective space  $\mathbb{C}\mathbb{P}^n$  we had a CW decomposition with one  $2i$ -cell for each  $0 \leq i \leq n$ . Thus, for the homology we obtain

$$H_k(\mathbb{C}\mathbb{P}^n) \cong \begin{cases} \mathbb{Z} & , \quad k = 2l, 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Let us say that a CW complex has **dimension bounded by  $n$**  if it has no cells in dimension larger than  $n$ . Using this terminology, the following is immediate for a CW complex  $X$ :

- (1) If  $\dim(X) \leq n$ , then  $H_k(X) \cong 0$  for all  $k > n$ .
- (2) If  $X$  is dimension-wise finite, then all  $H_k(X)$  are finitely generated.

For our further study of the homology of CW complexes let us recall the following two results.

**Lemma 2.** *Let  $X$  be obtained from  $A$  by attaching an  $n$ -cell along  $f: \partial e^n \rightarrow A$ ,  $X = A \cup_f e^n$ . Then*

$$H_k(X, A) \cong \begin{cases} \mathbb{Z} & , \quad k = n \\ 0 & , \quad \text{otherwise.} \end{cases}$$

*Moreover, the attaching map applied to any orientation class of  $(e^n, \partial e^n)$  gives us a generator of  $H_n(X, A)$ .*

Using this lemma, we can draw some consequences for the homology of CW complexes by ‘induction on the number of cells’. In the case of finitely many  $n$ -cells the following result was already established in the previous lecture. We leave it to the reader to deduce the general case from this using filtered colimits.

**Proposition 3.** *For any CW complex  $X$ ,  $H_k(X^{(n)}, X^{(n-1)}) \cong 0$  for all  $k \neq n$ .*

We now continue establishing some interesting facts about the singular homology of CW complexes.

**Proposition 4.** *For any CW-complex  $X$  and any  $n \geq 0$  we have  $H_i(X, X^{(n)}) \cong 0$ , for  $i \leq n$ .*

*Proof.* It suffices to prove this result for finite CW complexes  $X$ . The general case will follow by an argument using filtered colimits. The proof will be by induction over the number of cells in  $X$ . If  $X$  has dimension 0, the assertion is clear. Let us suppose that the proposition holds for  $A$ , and let us consider  $X = A \cup e^k$ . Then by excision as in the previous lecture, if  $n \geq k$  then

$$H_i(X, X^{(n)}) \cong H_i(A, A^{(n)}).$$

If  $k > n$  let us again consider the long exact sequence in singular homology associated to the triple

$$A^{(n)} = X^{(n)} \xrightarrow{\subseteq} A \xrightarrow{\subseteq} A \cup e^k = X,$$

a part of which looks like:

$$\dots \rightarrow H_i(A, A^{(n)}) \rightarrow H_i(X, A^{(n)}) = H_i(X, X^{(n)}) \rightarrow H_i(X, A) \rightarrow \dots$$

By Lemma 2 the group  $H_i(X, A)$  is nonzero only for  $i = k > n$ . Moreover, by induction the group  $H_i(A, A^{(n)})$  is zero for  $i \leq n$ . So surely the group in the middle is zero for  $i \leq n$  as intended.  $\square$

Let us now show that the range in which the singular homology of a CW complex is possibly nontrivial is bounded by its dimension.

**Proposition 5.** *Let  $X$  be a CW complex of dimension  $\leq n$ . Then  $H_i(X) \cong 0$  for  $i > n$ .*

*Proof.* Again, we prove the result for finite CW complexes by induction on the number of cells. The case of an infinite CW complex will follow by a colimit-argument. If  $\dim(X) = 0$  then the proposition is clear. Suppose the proposition holds for  $A$ , and let  $X = A \cup e^k$ , so in particular  $k \leq n$ . A typical part of the long exact homology sequence of the pair  $(X, A)$  looks like:

$$\dots \rightarrow H_i(A) \rightarrow H_i(X) \rightarrow H_i(X, A) \rightarrow \dots$$

Now the group  $H_i(X, A)$  is trivial for  $i \neq k$  (so surely for  $i > n$ ) as is the group  $H_i(A)$  for  $i > n$  by induction assumption. Thus also  $H_i(X) \cong 0$  for  $i > n$  concluding the proof.  $\square$

We can now prove a more explicit addendum to Proposition 3. If  $X$  is a CW complex, let us choose for each  $n$ -cell a characteristic map  $f$  and a homeomorphism  $\alpha$  as in

$$\Delta^n \xrightarrow{\alpha} e^n \xrightarrow{f} X.$$

Then  $f \circ \alpha$  is a cycle in  $C_n(X^{(n)}, X^{(n-1)})$ , so we obtain a homology class  $[f \circ \alpha] \in H_n(X^{(n)}, X^{(n-1)})$ . Doing this for each  $n$ -cell gives a well-defined homomorphism

$$\phi_n: C_n^{\text{cell}}(X) \rightarrow H_n(X^{(n)}, X^{(n-1)}).$$

It might be helpful to refamiliarize yourself with the proof of Proposition 3 (as given in the previous lecture) before reading the proof of the following proposition.

**Proposition 6.** *For all CW complexes  $X$  and all  $n$ , the map  $\phi_n: C_n^{\text{cell}}(X) \rightarrow H_n(X^{(n)}, X^{(n-1)})$  is an isomorphism.*

*Proof.* Again, it suffices to prove this for finite CW complexes  $X$ , the case where  $X$  has dimension zero is clear, and we consider only the induction step  $X = A \cup e^k$ . If  $k \neq n$  then

$$H_n(X^{(n)}, X^{(n-1)}) \cong H_n(A^{(n)}, A^{(n-1)})$$

(as we saw in the proof of Proposition 3 in the previous lecture) and also  $C_n^{\text{cell}}(A) = C_n^{\text{cell}}(X)$ . So we only need to look at the case  $k = n$ . But here we have a commutative diagram of the following form

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(A^{(n)}, A^{(n-1)}) & \longrightarrow & H_n(X^{(n)}, X^{(n-1)}) & \longrightarrow & H_n(X^{(n)}, A^{(n)}) & \longrightarrow & 0 \\ & & \uparrow \phi_n & & \uparrow \phi_n & & \uparrow \cong & & \\ 0 & \longrightarrow & C_n^{\text{cell}}(A) & \longrightarrow & C_n^{\text{cell}}(X) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

It is obvious that the last row is exact but also the first row is exact: for this consider the long exact sequence associated to the triple

$$A^{(n-1)} = X^{(n-1)} \xrightarrow{\subseteq} A^{(n)} \xrightarrow{\subseteq} X^{(n)}$$

and use that both groups  $H_{n+1}(X^{(n)}, X^{(n-1)})$  and  $H_{n-1}(A^{(n)}, A^{(n-1)})$  vanish. The fact that this diagram commutes follows from the explicit description of the isomorphism  $\mathbb{Z} \cong H_n(X^{(n)}, A^{(n)})$ . But by our induction assumption the vertical map on the left is an isomorphism. Thus we can deduce by the 5-lemma that also the vertical map in the middle is an isomorphism, completing the induction step.  $\square$

Thus, these relative homology groups are just free abelian groups generated by the various indexing sets of the cell structure. We now want to show that these relative homology groups themselves assemble into a chain complex, and in the next lecture we show that the homology of this new complex again calculates the homology of the space. A priori this does not seem to be an efficient idea: we build a complex consisting of relative homology groups of a space in order to calculate the homology groups of that same space. However, as we saw these relative homology groups have an easy explicit description and we will see that this alternative way of calculating the homology is very convenient. This is also due to the fact that the differential can be given in quite explicit geometric terms. If one has a good understanding of the attaching maps of a given CW complex, then this allows for the calculation of its homology.

Here, we give an abstract description of the differentials. The translation into more geometric terms will be given in the following lecture. Let us recall that associated to a *triple of spaces*  $(X, A, B)$  there is a *connecting homomorphism*

$$\Delta_n: H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \rightarrow H_{n-1}(A, B),$$

were  $\delta$  is the connecting homomorphism of the pair  $(X, A)$  and the undecorated morphism belongs to the long exact sequence of the pair  $(A, B)$ .

Let  $X$  be a CW complex. For each  $n \geq 1$  there is the triple of spaces  $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$  (we use the standard convention  $X^{(-1)} = \emptyset$ ). Let us denote the connecting homomorphism of this triple by

$$\partial_n^{\text{cell}}: H_n(X^{(n)}, X^{(n-1)}) \rightarrow H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

A key property of these maps is given by the following lemma.

**Lemma 7.** *For a CW complex  $X$  and  $n \geq 2$  we have*

$$0 = \partial_{n-1}^{\text{cell}} \circ \partial_n^{\text{cell}}: H_n(X^{(n)}, X^{(n-1)}) \rightarrow H_{n-2}(X^{(n-2)}, X^{(n-3)}).$$

*Proof.* This follows since the composition of these cellular boundary homomorphisms is given by

$$\begin{array}{ccccc} H_n(X^{(n)}, X^{(n-1)}) & & & & \\ \delta \downarrow & & & & \\ H_{n-1}(X^{(n-1)}) & \longrightarrow & H_{n-1}(X^{(n-1)}, X^{(n-2)}) & & \\ & \searrow & \delta \downarrow & & \\ & & 0 & \longrightarrow & H_{n-2}(X^{(n-2)}, X^{(n-3)}). \end{array}$$

But the composition of the second and the third morphism is trivial since these are two subsequent morphisms belonging to the long exact homology sequence of the pair  $(X^{(n-1)}, X^{(n-2)})$ .  $\square$

From now on we will use these relative homology groups as *definitions* of  $C_{\bullet}^{\text{cell}}(X)$ , but keep in mind that these are isomorphic to the groups described at the beginning of this lecture. Thus we make the following definition.

**Definition 8.** The **cellular chain complex**  $C_{\bullet}^{\text{cell}}(X)$  of a CW complex  $X$  is given by the **cellular chain groups**

$$C_n^{\text{cell}}(X) = H_n(X^{(n)}, X^{(n-1)}), \quad n \geq 0,$$

together with the **cellular boundary homomorphisms**

$$\partial_n^{\text{cell}}: C_n^{\text{cell}}(X) = H_n(X^{(n)}, X^{(n-1)}) \rightarrow C_{n-1}^{\text{cell}}(X) = H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

The **cellular homology**  $H_n^{\text{cell}}(X)$  of  $X$  is given by

$$H_n^{\text{cell}}(X) = H_n(C_{\bullet}^{\text{cell}}(X)), \quad n \geq 0.$$

Note that cellular homology is functorial with respect to *cellular maps* of CW complexes. This follows from the naturality of the connecting homomorphism of a triple since any cellular map  $f: X \rightarrow Y$  induces maps of triples

$$f: (X^{(n)}, X^{(n-1)}, X^{(n-2)}) \rightarrow (Y^{(n)}, Y^{(n-1)}, Y^{(n-2)}).$$

Thus, cellular homology defines a functor on the category of CW complexes and cellular maps. In the next lecture, we will show that this functor is naturally isomorphic to the singular homology functor.