

MATH 215B HOMEWORK 6 SOLUTIONS

1. Let X be a Moore space $M(\mathbb{Z}/m, n)$ obtained from S^n by attaching a cell e^{n+1} by a map of degree m .

- (a) (6 marks) Show that the quotient map $X \rightarrow X/S^n = S^{n+1}$ induces the trivial map on $\tilde{H}_i(-; \mathbb{Z})$ for all i , but not on $H^{n+1}(-; \mathbb{Z})$. Deduce that the splitting in the universal coefficient theorem for cohomology cannot be natural.
- (b) (4 marks) Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on $\tilde{H}^i(-; \mathbb{Z})$ for all i , but not on $H_n(-; \mathbb{Z})$.

Solution

- (a) Recall that $\tilde{H}_i(M(\mathbb{Z}/m, n))$ is trivial except when $i = n$, in which case it is $\mathbb{Z}/m$. So for every i , either $H_i(X)$ or $H_i(S^{n+1})$ is trivial. So of course the quotient map $X \rightarrow S^{n+1}$ induces trivial maps on homology.

As for cohomology, consider the long exact sequence of the pair (X, S^n) . Part of it looks like this:

$$\tilde{H}^{n+1}(X/S^n) \rightarrow \tilde{H}^{n+1}(X) \rightarrow \tilde{H}^{n+1}(S^n)$$

The right group is trivial. By the universal coefficient theorem, $\tilde{H}^{n+1}(X) \approx \text{Ext}(\mathbb{Z}/m, \mathbb{Z}) \approx \mathbb{Z}/m$. So we have

$$\mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0$$

so the map is surjective, not trivial.

Now consider the maps $X \rightarrow S^{n+1}$, and the homomorphisms this induces in the universal coefficient theorem:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}(H_n(S^{n+1}), \mathbb{Z}) & \longrightarrow & H^{n+1}(S^{n+1}) & \longrightarrow & \text{Hom}(H_{n+1}(S^{n+1}), \mathbb{Z}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ext}(H_n(X), \mathbb{Z}) & \longrightarrow & H^{n+1}(X) & \longrightarrow & \text{Hom}(H_{n+1}(X), \mathbb{Z}) \longrightarrow 0 \end{array}$$

which we calculate as

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \overset{\leftarrow}{\rightleftarrows} & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}/m & \longrightarrow & \mathbb{Z}/m & \overset{\leftarrow}{\rightleftarrows} & 0 \longrightarrow 0 \end{array}$$

We've drawn in the splittings. If they were natural, then the square

$$\begin{array}{ccc} \mathbb{Z} & \overset{\leftarrow}{\rightleftarrows} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}/m & \overset{\leftarrow}{\rightleftarrows} & 0 \end{array}$$

would commute. But it cannot commute because the left homomorphism is not trivial.

- (b) The inclusion $S^n \hookrightarrow X$ induces trivial maps on $\tilde{H}^i(-; \mathbb{Z})$ for dimension reasons: $\tilde{H}^i(S^n)$ is trivial for $i \neq n$, and $\tilde{H}^i(X)$ is trivial for $i \neq n+1$.

As for homology, consider the long exact sequence of the pair (X, S^n) . Part of it looks like

$$\tilde{H}_n(S^n) \rightarrow \tilde{H}_n(X) \rightarrow \tilde{H}_n(X/S^n)$$

The last group is isomorphic to $\tilde{H}^n(S^{n+1}) = 0$. So we have

$$\mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0$$

the first map of which is surjective and thus nontrivial.

- 2.** (12 marks) Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori, shown below.

Solution

Recall that the homology of M_g is as follows:

$$H_i(M_g) \approx \begin{cases} \mathbb{Z} & i = 0 \\ \oplus_{2g} \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

H_1 is generated by g “longitudinal” classes $a_1, \dots, a_g$ and g “latitudinal” classes $b_1, \dots, b_g$. H_2 is generated by a single class c . On the other hand, the homology of the wedge sum of g tori is:

$$H_i(\vee_g(S^1 \times S^1)) \approx \begin{cases} \mathbb{Z} & i = 0 \\ \oplus_{2g} \mathbb{Z} & i = 1 \\ \oplus_g \mathbb{Z} & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

H_1 is generated by g longitudinal classes and g latitudinal classes, $a_1, \dots, a_g$ and $b_1, \dots, b_g$ respectively. H_2 is generated by the classes $c_1, \dots, c_g$, one for each wedge summand.

I claim that the quotient map in the problem statement induces maps on homology as follows:

$$\begin{aligned} H_*(M_g) &\longrightarrow H_*(\vee_g(S^1 \times S^1)) \\ a_i &\longmapsto a_i \\ b_i &\longmapsto b_i \\ c &\longmapsto \sum c_i \end{aligned}$$

You can see why this is so for the 1-dimensional generators by choosing singular cycles representing the generators. As for the 2-dimensional generators, we can

consider local homology groups at a point $x \in M_g$ such that $q(x)$ is not the base-point:

$$\begin{array}{ccc}
 H_2(M_g) & \xrightarrow{q_*} & H_2(\vee_g(S^1 \times S^1)) \\
 \downarrow \approx & & \downarrow \approx \\
 H_2(M_g, M_g - x) & \xrightarrow{q_*} & H_2(\vee_g(S^1 \times S^1), \vee_g(S^1 \times S^1) - q(x)) \\
 \downarrow \approx & & \downarrow \approx \\
 \tilde{H}_2(S^2) & \xrightarrow{\approx} & \tilde{H}_2(S^2)
 \end{array}$$

The upper vertical maps come from the long exact sequence of a pair, and the lower two maps are the isomorphisms $H_i(X, A) \approx \tilde{H}_i(X/A)$. q is a local homeomorphism at x and so sends a generator of $H_2(M_g, M_g - x)$ (such as the image of c) to a generator of $H_2(\vee_g(S^1 \times S^1), \vee_g(S^1 \times S^1) - q(x))$. If $q(x)$ lies in the i^{th} torus, then this latter group is isomorphic to $H_2(S^1 \times S^1, S^1 \times S^1 - q(x))$ which is generated by the image of c_i . Since this holds for all i , we conclude that in the top row, c is sent to a sum $\sum \pm c_i$. If we chose the generators c_i correctly, the signs will all be positive.

When we apply the universal coefficient theorem to q_* , we find that all the Ext terms vanish. So the cohomology groups are the duals of the homology groups: $H^*(M_g)$ is free in each dimension with generators α_i, β_i in dimension 1 dual to a_i, b_i and generator γ in dimension 2 dual to c . $H^*(\vee_g(S^1 \times S^1))$ is free in each dimension with generators $\alpha_i, \beta_i, \gamma_i$ dual to a_i, b_i, c_i . And of course, one generator for each in dimension 0.

Also, the universal coefficient theorem is natural, so q^* is the dual of q_* :

$$\begin{array}{ccc}
 H^*(M_g) & \xleftarrow{q^*} & H^*(\vee_g(S^1 \times S^1)) \\
 \alpha_i & \longleftarrow & \alpha_i \\
 \beta_i & \longleftarrow & \beta_i \\
 \gamma & \longleftarrow & \gamma_i
 \end{array}$$

Recall that the reduced cohomology ring of the wedge sum of spaces is the product of the reduced cohomology rings: $\tilde{H}^*(\vee_g(S^1 \times S^1)) \approx \prod_g \tilde{H}^*(S^1 \times S^1)$. (These rings don't have identities, so you may prefer to call them $\mathbb{Z}$ -algebras.) So, using the cup product structure of $H^*(S^1 \times S^1)$, we learn that $\alpha_i \cup \beta_i = \gamma_i$ and $\beta_i \cup \alpha_i = -\gamma_i$, and all other products of the 1-dimensional generators are trivial. This determines the cup product structure of $H^*(\vee_g(S^1 \times S^1))$.

Now using q^* , we can see that in $H^*(M_g)$, $\alpha_i \cup \beta_i = q^*(\alpha_i) \cup q^*(\beta_i) = q^*(\alpha_i \cup \beta_i) = q^*(\gamma_i) = \gamma$. Similarly, we find that

$$\begin{array}{ll}
 \beta_i \cup \alpha_i = -\gamma & \\
 \alpha_i \cup \beta_j = \beta_j \cup \alpha_i = 0 & \text{for } i \neq j \\
 \alpha_i \cup \alpha_j = 0 & \\
 \beta_i \cup \beta_j = 0 &
 \end{array}$$

3. (10 marks) Using the cup product $H^k(X, A; R) \times H^l(X, B; R) \rightarrow H^{k+l}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B , then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. This applies in particular if X is a suspension. Generalize to the situation that X is the union of n contractible open subsets, to show that all n -fold cup products of positive-dimensional classes are zero.

Solution

Since A is contractible, the long exact sequence for the pair (X, A) gives us that the maps $j^* : H^k(X, A; R) \rightarrow H^k(X; R)$ are isomorphisms for all $k \geq 1$. Same thing for the maps $i^* : H^k(X, B; R) \rightarrow H^k(X; R)$.

Let us study how j^* and i^* behave with respect to cup products. Consider the following two diagrams of projections

$$\begin{array}{ccccc} C_n(X) & \xrightarrow{j} & C_n(X)/C_n(A) & \xrightarrow{f_1} & C_n(X)/C_n(A+B) \\ & & & & \\ C_n(X) & \xrightarrow{i} & C_n(X)/C_n(B) & \xrightarrow{f_2} & C_n(X)/C_n(A+B) \end{array}$$

whose composition are both the projection $g : C_n(X) \rightarrow C_n(X)/C_n(A+B)$. Then given $a \in H^*(X, A; R)$ and $b \in H^*(X, B; R)$, I will show that $j^*(a) \cup i^*(b) = g^*(a \cup b)$, where the inner cup product is the relative cup product $H^*(X, A; R) \times H^*(X, B; R) \rightarrow H^*(X, A+B; R)$ and the outer denotes the cup product $H^*(X; R) \times H^*(X; R) \rightarrow H^*(X; R)$.

The relative cup product is defined on two cochains $\phi : C_k(X)/C_k(A) \rightarrow R$ and $\psi : C_l(X)/C_l(B) \rightarrow R$ by first taking $(\phi \circ j) \cup (\psi \circ i) : C_{k+l}(X) \rightarrow R$. Then $(\phi \cup \psi)([w]) = ((\phi \circ j) \cup (\psi \circ i))(w)$ for $w \in C_n(X)$, where I am denoting by $[w]$ its class in $C_{k+l}(X)/C_{k+l}(A+B)$, that is, $[w] = g(w)$. This is well defined because $\phi \circ j$ takes elements of $C_n(A)$ to 0 and $\psi \circ i$ takes elements of $C_n(B)$ to 0.

$$g^\#(\phi \cup \psi)(w) = (\phi \cup \psi)(g(w)) = (\phi \cup \psi)([w]) = ((\phi \circ j) \cup (\psi \circ i))(w) = (j^\#(\phi) \cup i^\#(\psi))(w)$$

So $g^*(\phi \cup \psi) = j^*(\phi) \cup i^*(\psi)$. Since cup products are bilinear, this shows the result for general elements of $H^*(X, A; R)$ and $H^*(X, B; R)$.

Let $x \in H^k(X; R)$, $y \in H^l(X; R)$ with $k, l > 0$. Then there are $z \in H^k(X, A; R)$ and $w \in H^l(X, B; R)$ with $j^*(z) = x$ and $i^*(w) = y$.

$$x \cup y = j^*(z) \cup i^*(w) = g^*(z \cup w) = 0$$

because $H^*(X, A+B; R) \cong H^*(X, A \cup B; R) = H^*(X, X; R) = 0$. Excision was used in the first isomorphism.

When X is the suspension of Y , we can write X as the union of two cones on Y . Each cone has an open neighbourhood that deformation retracts onto the cone and so it is contractible.

Assume now that X is the union of n contractible open subsets $U_1, \dots, U_n$. Define a relative cup product

$$H^*(X, U_1; R) \times \dots \times H^*(X, U_n; R) \rightarrow H^*(X, U_1 \cup \dots \cup U_n; R)$$

by setting $(\phi_1 \cup \dots \cup \phi_n)([w]) = ((\phi_1 \circ p_1) \cup \dots \cup (\phi_n \circ p_n))(w)$, where $[w] = g(w)$ denotes the class of $w \in C_m(X)$ in $C_m(X)/C_m(U_1 + \dots + U_n)$ and $p_j : C_m(X) \rightarrow C_m(X)/C_m(U_j)$ are the projections. It is obvious now that $g^*(\phi_1 \cup \dots \cup \phi_n) = p_1^*(\phi_1) \cup \dots \cup p_n^*(\phi_n)$. Now given elements $x_1, \dots, x_n$ in the cohomology of X with positive degree, we have that $x_j = p_j^*(y_j)$ for some y_j in $H^*(X, U_j; R)$ because the U_j are contractible and so:

$$x_1 \cup \dots \cup x_n = p_1^*(y_1) \cup \dots \cup p_n^*(y_n) = g^*(y_1 \cup \dots \cup y_n) = 0$$

because $H^*(X, U_1 + \dots + U_n; R) \cong H^*(X, U_1 \cup \dots \cup U_n; R) = H^*(X, X; R) = 0$. Again we used excision in the first isomorphism.

4. (a) (6 marks) Using the cup product structure, show that there is no map $\mathbb{RP}^n \rightarrow \mathbb{RP}^m$ inducing a nontrivial map $H^1(\mathbb{RP}^m; \mathbb{Z}_2) \rightarrow H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ if $n > m$. What is the corresponding result for maps $\mathbb{CP}^n \rightarrow \mathbb{CP}^m$?

(b) (8 marks) Prove the Borsuk-Ulam theorem by the following argument. Suppose on the contrary that $f : S^n \rightarrow \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all x . Then define $g : S^n \rightarrow S^{n-1}$ by $g(x) = (f(x) - f(-x)) / |f(x) - f(-x)|$, so $g(-x) = -g(x)$ and g induces a map $\mathbb{RP}^n \rightarrow \mathbb{RP}^{n-1}$. Show that part (a) applies to this map.

Solution

(a) Recall that for any n , $H^*(\mathbb{RP}^n; \mathbb{Z}/2) \approx \mathbb{Z}/2[\alpha]/(\alpha^{n+1})$, with $|\alpha| = 1$. If $f : \mathbb{RP}^n \rightarrow \mathbb{RP}^m$ is a map that induces a nontrivial map on $H^1(-; \mathbb{Z}/2)$, that means that $f^*(\alpha) = \alpha$. Therefore $f^*(\alpha^{m+1}) = f^*(\alpha^m)f^*(\alpha) = \alpha^m\alpha = \alpha^{m+1}$, which is a nontrivial element in $H^{m+1}(\mathbb{RP}^n; \mathbb{Z}/2)$. But this is impossible because α^{m+1} is trivial in $H^*(\mathbb{RP}^m; \mathbb{Z}/2)$.

Recall that for any n , $H^*(\mathbb{CP}^n; \mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^{n+1})$, with $|\alpha| = 2$. A similar argument to the above paragraph shows that there is no map $\mathbb{CP}^n \rightarrow \mathbb{CP}^m$ inducing a nontrivial map $H^2(\mathbb{CP}^m) \rightarrow H^2(\mathbb{CP}^n)$ if $n > m$. (The argument is as follows: $f^*(\alpha^{m+1}) = f^*(\alpha^m)f^*(\alpha) = \alpha^m\alpha = \alpha^{m+1}$, which is a nontrivial element in $H^{m+1}(\mathbb{CP}^n)$. But α^{m+1} is trivial in $H^*(\mathbb{CP}^m)$.)

(b) If we have f as in the problem statement, then g is well-defined and continuous because the denominator never vanishes. Also note that the norm of $g(x)$ is 1, so $g(x)$ lies in the sphere S^{n-1} ; and note also that $g(-x) = -g(x)$. Thus g descends to a map of projective spaces as follows:

$$\begin{array}{ccc} S^n & \xrightarrow{g} & S^{n-1} \\ \downarrow & & \downarrow \\ \mathbb{RP}^n & \xrightarrow{g} & \mathbb{RP}^{n-1} \end{array}$$

The vertical maps are the usual two-sheeted covering maps, and we will call the lower map g as well.

We use the theory of covering spaces: Suppose we have a based loop in $\mathbb{RP}^n$ that represents a generator of π_1 . It lifts to a path in its universal cover S^n from a point x to its antipode $-x$. The image of that path under g is a path from $g(x)$ to its antipode $g(-x) = -g(x)$. The image of this

path under the covering map is a loop that is not nullhomotopic. So we have that $g_* : \pi_1(\mathbb{RP}^n) \rightarrow \pi_1(\mathbb{RP}^{n-1})$ is nontrivial.

If $n = 2$, this is already a contradiction because there is no nontrivial homomorphism $\mathbb{Z}/2 \rightarrow \mathbb{Z}$. If $n = 1$, then $\mathbb{RP}^0$ is a point, and we have a contradiction because there is no nontrivial homomorphism $\mathbb{Z} \rightarrow 0$. From now on we assume $n > 2$. In particular, the nontrivial homomorphism g_* is in fact an isomorphism.

Recall from Section 2.A in Hatcher that H_1 is the abelianization of π_1 . In fact there is a homomorphism $h : \pi_1(X) \rightarrow H_1(X)$ which is an isomorphism if X is path-connected and $\pi_1(X)$ is abelian. This homomorphism is natural, meaning that we have a commutative square:

$$\begin{array}{ccc} \pi_1(\mathbb{RP}^n) & \xrightarrow[\approx]{g_*} & \pi_1(\mathbb{RP}^{n-1}) \\ \approx \downarrow h & & \approx \downarrow h \\ H_1(\mathbb{RP}^n) & \xrightarrow{g_*} & H_1(\mathbb{RP}^{n-1}) \end{array}$$

The top map is an isomorphism, so the bottom map is also. Now we apply the universal coefficient theorem. Since H_0 is free, the Ext terms vanish, and we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathbb{RP}^{n-1}; \mathbb{Z}/2) & \longrightarrow & \text{Hom}(H_1(\mathbb{RP}^{n-1}), \mathbb{Z}/2) & \longrightarrow & 0 \\ & & \downarrow g^* & & \downarrow (g_*)^* & & \\ 0 & \longrightarrow & H^1(\mathbb{RP}^n; \mathbb{Z}/2) & \longrightarrow & \text{Hom}(H_1(\mathbb{RP}^n), \mathbb{Z}/2) & \longrightarrow & 0 \end{array}$$

Since g_* is an isomorphism, its dual $(g_*)^*$ is an isomorphism too, and so g^* is an isomorphism, taking the generator α to the generator α . This violates part (a), so we have a contradiction.

5. (6 marks) Use cup products to show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$.

Solution

The cohomology ring of $\mathbb{RP}^3$ with $\mathbb{Z}/2$ coefficients is $\mathbb{Z}[\alpha]/(\alpha^4)$, with $|\alpha| = 1$. In particular, it is a free $\mathbb{Z}/2$ -module in each dimension, with generators $1, \alpha, \alpha^2, \alpha^3$ of dimensions 0, 1, 2, and 3, respectively.

The reduced cohomology ring (or $\mathbb{Z}/2$ -algebra, if you like) of $\mathbb{RP}^2 \vee S^3$ with coefficients in $\mathbb{Z}/2$ is $\tilde{H}^*(\mathbb{RP}^2; \mathbb{Z}/2) \times \tilde{H}^*(S^3; \mathbb{Z}/2)$. So the unreduced cohomology ring of this space is also a $\mathbb{Z}/2$ -module in each dimension, with generators $1, \alpha, \alpha^2, s$ of dimensions 0, 1, 2, and 3, respectively. But unlike the case of $\mathbb{RP}^3$, we have $\alpha^3 = 0$. So the cohomology rings are not isomorphic (as rings), and so the spaces are not homotopy equivalent.

6. (10 marks) Show that the spaces $(S^1 \times \mathbb{CP}^\infty)/(S^1 \times \{x_0\})$ and $S^3 \times \mathbb{CP}^\infty$ have isomorphic cohomology rings with $\mathbb{Z}$ or any other coefficients.

Solution

Let $X = (S^1 \times \mathbb{CP}^\infty)/(S^1 \times \{x_0\})$ for short.

Note that the proof of Theorem 3.12 for $\mathbb{CP}^\infty$ works for any coefficients R , so $H^*(\mathbb{CP}^\infty; R) \cong R[a_2]$. Alternatively, since $\mathbb{CP}^n$ is orientable, then it is R -orientable

and we can use the same argument given in class using Corollary 3.39. Also, note that since the homology of S^n is free abelian, all the Ext terms in the universal coefficient theorem for cohomology vanish and so $H^n(S^n; R) = \text{Hom}(H_n(S^n), R) \cong R$. Therefore, we can use Künneth theorem for products of spaces with spheres. So we have:

$$H^*(S^3 \times \mathbb{C}P^\infty; R) \cong H^*(S^3; R) \otimes_R H^*(\mathbb{C}P^\infty; R) \cong \Lambda_R[b_3] \otimes R[a_2]$$

On the other hand, by the relative Künneth theorem,

$$H^*(S^1 \times \mathbb{C}P^\infty, S^1 \times \{x_0\}; R) \cong H^*(S^1; R) \otimes H^*(\mathbb{C}P^\infty, x_0; R) \cong \Lambda_R[c_1] \otimes_R \tilde{R}[d_2]$$

where $\tilde{R}[d_2]$ is the polynomial ring $R[d]$ without the elements of degree 0. Since $(S^1 \times \mathbb{C}P^\infty, S^1 \times \{x_0\})$ is a CW pair, it is a good pair and so $H^*(S^1 \times \mathbb{C}P^\infty, S^1 \times \{x_0\}; R) = \tilde{H}^*(X; R)$. Then if $k > 0$, then $H^{2k}(X; R)$ is generated as an R -module by d_2^k , $H^1(X; R) = 0$ and $H^{2k+1}(X; R)$ is generated as an R -module by $c_1 \cup d_2^k$ for $k > 0$. Now $H^*(X; R)$ is the reduced cohomology plus the elements in degree 0 that act as multiplication by elements of R , so $H^*(X; R)$ is generated multiplicatively by the elements $1 \in H^0(X; R)$, $d_2 \in H^2(X; R)$ and $c_1 \cup d_2 \in H^3(X; R)$.

Let $\phi : \Lambda_R[b_3] \otimes_R R[a_2] \rightarrow H^*(X; R)$ be the only ring homomorphism that takes 1 to 1, b_3 to $c_1 \cup d_2$ and a_2 to d_2 . This map has as its inverse the only ring homomorphism $\psi : H^*(X; R) \rightarrow \Lambda_R[b_3] \otimes_R R[a_2]$ that takes 1 to 1, $c_1 \cup d_2$ to b_3 and d_2 to a_2 . These two maps are well defined because:

$$(c_1 \cup d_2)^2 = \pm c_1^2 \cup d_2 = 0 = b_3^2$$

$$b_3 \cup a_2 = (-1)^6 a_2 \cup b_3 = a_2 \cup b_3$$

$$(c_1 \cup d_2) \cup d_2 = (-1)^6 d_2 \cup (c_1 \cup d_2) = d_2 \cup (c_1 \cup d_2)$$

and these are the only relations in these rings. In fact, we did not need to look at the last two relations since they would be automatically preserved by the fact that the products in the cohomology ring are graded commutative and the maps ϕ and ψ were graded maps by definition.