Course 421: Algebraic Topology
Section 6: Simplicial Homology Groups

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Contents

6 Simplicial Homology Groups

6.1 The Chain Groups of a Simplicial Complex

Let K be a simplicial complex. For each non-negative integer q, let $\Delta_q(K)$ be the additive group consisting of all formal sums of the form

$$n_1(\mathbf{v}_0^1, \mathbf{v}_1^1, \ldots, \mathbf{v}_q^1) + n_2(\mathbf{v}_0^2, \mathbf{v}_1^2, \ldots, \mathbf{v}_q^2) + \cdots + n_s(\mathbf{v}_0^s, \mathbf{v}_1^s, \ldots, \mathbf{v}_q^s),$$

where $n_1, n_2, \ldots, n_s$ are integers and $\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r$ are (not necessarily distinct) vertices of K that span a simplex of K for $r = 1, 2, \ldots, s$. (In more formal language, the group $\Delta_q(K)$ is the *free Abelian group* generated by the set of all $(q+1)$-tuples of the form $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.)

We recall some basic facts concerning *permutations*. A *permutation* of a set S is a bijection mapping S onto itself. The set of all permutations of some set S is a group; the group multiplication corresponds to composition of permutations. A *transposition* is a permutation of a set S which interchanges two elements of S, leaving the remaining elements of the set fixed. If S is finite and has more than one element then any permutation of S can be expressed as a product of transpositions. In particular any permutation of the set $\{0, 1, \ldots, q\}$ can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ and j for some j.

Associated to any permutation π of a finite set S is a number ϵ_π, known as the *parity* or *signature* of the permutation, which can take on the values ± 1. If π can be expressed as the product of an even number of transpositions, then $\epsilon_\pi = +1$; if π can be expressed as the product of an odd number of transpositions then $\epsilon_\pi = -1$. The function $\pi \mapsto \epsilon_\pi$ is a homomorphism from the group of permutations of a finite set S to the multiplicative group $\{+1, -1\}$ (i.e., $\epsilon_{\pi\rho} = \epsilon_\pi \epsilon_\rho$ for all permutations π and ρ of the set S). Note in particular that the parity of any transposition is -1.

Definition The qth *chain group* $C_q(K)$ of the simplicial complex K is defined to be the quotient group $\Delta_q(K)/\Delta_q^0(K)$, where $\Delta_q^0(K)$ is the subgroup of $\Delta_q(K)$ generated by elements of the form $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct, and by elements of the form

$$(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)}) - \epsilon_\pi(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$$

where π is some permutation of $\{0, 1, \ldots, q\}$ with parity ϵ_π. For convenience, we define $C_q(K) = \{0\}$ when $q < 0$ or $q > \dim K$, where $\dim K$ is the dimension of the simplicial complex K. An element of the chain group $C_q(K)$ is referred to as q-*chain* of the simplicial complex K.

We denote by $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ the element $\Delta_q^0(K) + (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of $C_q(K)$ corresponding to $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$. The following results follow immediately from the definition of $C_q(K)$.

Lemma 6.1 *Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices of a simplicial complex K that span a simplex of K. Then*

- $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = 0$ *if $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct,*
- $\langle \mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)} \rangle = \epsilon_\pi \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ *for any permutation π of the set $\{0, 1, \ldots, q\}$.*

Example If $\mathbf{v}_0$ and $\mathbf{v}_1$ are the endpoints of some line segment then

$$\langle \mathbf{v}_0, \mathbf{v}_1 \rangle = -\langle \mathbf{v}_1, \mathbf{v}_0 \rangle.$$

If $\mathbf{v_0}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vertices of a triangle in some Euclidean space then

$$\begin{aligned} \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \rangle &= \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0 \rangle = \langle \mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1 \rangle = -\langle \mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0 \rangle \\ &= -\langle \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_1 \rangle = -\langle \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2 \rangle. \end{aligned}$$

Definition An *oriented q-simplex* is an element of the chain group $C_q(K)$ of the form $\pm\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$, where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are distinct and span a simplex of K.

An oriented simplex of K can be thought of as consisting of a simplex of K (namely the simplex spanned by the prescribed vertices), together with one of two possible ‘orientations’ on that simplex. Any ordering of the vertices determines an orientation of the simplex; any even permutation of the ordering of the vertices preserves the orientation on the simplex, whereas any odd permutation of this ordering reverses orientation.

Any q-chain of a simplicial complex K can be expressed as a sum of the form

$$n_1\sigma_1 + n_2\sigma_2 + \cdots + n_s\sigma_s$$

where $n_1, n_2, \ldots, n_s$ are integers and $\sigma_1, \sigma_2, \ldots, \sigma_s$ are *oriented* q-simplices of K. If we reverse the orientation on one of these simplices σ_i then this reverses the sign of the corresponding coefficient n_i. If $\sigma_1, \sigma_2, \ldots, \sigma_s$ represent distinct simplices of K then the coefficients $n_1, n_2, \ldots, n_s$ are uniquely determined.

Example Let $\mathbf{v_0}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ be the vertices of a triangle in some Euclidean space. Let K be the simplicial complex consisting of this triangle, together with its edges and vertices. Every 0-chain of K can be expressed uniquely in the form

$$n_0\langle \mathbf{v}_0\rangle + n_1\langle \mathbf{v}_1\rangle + n_2\langle \mathbf{v}_2\rangle$$

for some $n_0, n_1, n_2 \in \mathbb{Z}$. Similarly any 1-chain of K can be expressed uniquely in the form

$$m_0\langle \mathbf{v}_1, \mathbf{v}_2\rangle + m_1\langle \mathbf{v}_2, \mathbf{v}_0\rangle + m_2\langle \mathbf{v}_0, \mathbf{v}_1\rangle$$

for some $m_0, m_1, m_2 \in \mathbb{Z}$, and any 2-chain of K can be expressed uniquely as $n\langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\rangle$ for some integer n.

Lemma 6.2 *Let K be a simplicial complex, and let A be an additive group. Suppose that, to each $(q+1)$-tuple $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of vertices spanning a simplex of K, there corresponds an element $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of A, where*

- $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$ *unless* $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ *are all distinct,*
- $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ *changes sign on interchanging any two adjacent vertices* $\mathbf{v}_{j-1}$ *and* $\mathbf{v}_j$.

Then there exists a well-defined homomorphism from $C_q(K)$ to A which sends $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle$ to $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. This homomorphism is uniquely determined.

Proof The given function defined on $(q+1)$-tuples of vertices of K extends to a well-defined homomorphism $\alpha\colon \Delta_q(K) \to A$ given by

$$\alpha\left(\sum_{r=1}^{s} n_r(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r)\right) = \sum_{r=1}^{s} n_r\alpha(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r)$$

for all $\sum_{r=1}^{s} n_r(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r) \in \Delta_q(K)$. Moreover $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in \ker\alpha$ unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct. Also

$$(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)}) - \varepsilon_\pi(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in \ker\alpha$$

for all permutations π of $\{0, 1, \ldots, q\}$, since the permutation π can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ with j for some j and leave the rest of the set fixed, and the parity ε_π of π is given by $\varepsilon_\pi = +1$ when the number of such transpositions is even, and by $\varepsilon_\pi = -1$ when the number of such transpositions is odd. Thus the generators of $\Delta_q^0(K)$ are contained in $\ker\alpha$, and hence $\Delta_q^0(K) \subset \ker\alpha$. The required homomorphism $\tilde{\alpha}\colon C_q(K) \to A$ is then defined by the formula

$$\tilde{\alpha}\left(\sum_{r=1}^{s} n_r\langle \mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r\rangle\right) = \sum_{r=1}^{s} n_r\alpha(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r). \quad \blacksquare$$

6.2 Boundary Homomorphisms

Let K be a simplicial complex. We introduce below *boundary homomorphisms* $\partial_q: C_q(K) \to C_{q-1}(K)$ between the chain groups of K. If σ is an oriented q-simplex of K then $\partial_q(\sigma)$ is a $(q-1)$-chain which is a formal sum of the $(q-1)$-faces of σ, each with an orientation determined by the orientation of σ.

Let σ be a q-simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. For each integer j between 0 and q we denote by $\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$ the oriented $(q-1)$-face

$$\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_q \rangle$$

of the simplex σ obtained on omitting $\mathbf{v}_j$ from the set of vertices of σ. In particular

$$\langle \hat{\mathbf{v}}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle \equiv \langle \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle, \qquad \langle \mathbf{v}_0, \ldots, \mathbf{v}_{q-1}, \hat{\mathbf{v}}_q \rangle \equiv \langle \mathbf{v}_0, \ldots, \mathbf{v}_{q-1} \rangle.$$

Similarly if j and k are integers between 0 and q, where $j < k$, we denote by

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots \mathbf{v}_q \rangle$$

the oriented $(q-2)$-face $\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_q \rangle$ of the simplex σ obtained on omitting $\mathbf{v}_j$ and $\mathbf{v}_k$ from the set of vertices of σ.

We now define a 'boundary homomorphism' $\partial_q: C_q(K) \to C_{q-1}(K)$ for each integer q. Define $\partial_q = 0$ if $q \leq 0$ or $q > \dim K$. (In this case one or other of the groups $C_q(K)$ and $C_{q-1}(K)$ is trivial.) Suppose then that $0 < q \leq \dim K$. Given vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ spanning a simplex of K, let

$$\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

Inspection of this formula shows that $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ changes sign whenever two adjacent vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are interchanged.

Suppose that $\mathbf{v}_j = \mathbf{v}_k$ for some j and k satisfying $j < k$. Then

$$\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle + (-1)^k \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle,$$

since the remaining terms in the expression defining $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ contain both $\mathbf{v}_j$ and $\mathbf{v}_k$. However $(\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q)$ can be transformed to $(\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q)$ by making $k - j - 1$ transpositions which interchange $\mathbf{v}_j$ successively with the vertices $\mathbf{v}_{j+1}, \mathbf{v}_{j+2}, \ldots, \mathbf{v}_{k-1}$. Therefore

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle = (-1)^{k-j-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

Thus $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$ unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct. It now follows immediately from Lemma 6.2 that there is a well-defined homomorphism $\partial_q: C_q(K) \to C_{q-1}(K)$, characterized by the property that

$$\partial_q \left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle\right) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Lemma 6.3 *$\partial_{q-1} \circ \partial_q = 0$ for all integers q.*

Proof The result is trivial if $q < 2$, since in this case $\partial_{q-1} = 0$. Suppose that $q \geq 2$. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of K. Then

$$\begin{aligned} \partial_{q-1}\partial_q \left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle\right) &= \sum_{j=0}^{q} (-1)^j \partial_{q-1} \left(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle\right) \\ &= \sum_{j=0}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\ &\quad + \sum_{j=0}^{q} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle \\ &= 0 \end{aligned}$$

(since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1} \circ \partial_q$ is determined by its values on all oriented q-simplices of K. ∎

6.3 The Homology Groups of a Simplicial Complex

Let K be a simplicial complex. A q-chain z is said to be a *q-cycle* if $\partial_q z = 0$. A q-chain b is said to be a *q-boundary* if $b = \partial_{q+1} c'$ for some $(q+1)$-chain c'. The group of q-cycles of K is denoted by $Z_q(K)$, and the group of q-boundaries of K is denoted by $B_q(K)$. Thus $Z_q(K)$ is the kernel of the boundary homomorphism $\partial_q: C_q(K) \to C_{q-1}(K)$, and $B_q(K)$ is the image of the boundary homomorphism $\partial_{q+1}: C_{q+1}(K) \to C_q(K)$. However $\partial_q \circ \partial_{q+1} = 0$, by Lemma 6.3. Therefore $B_q(K) \subset Z_q(K)$. But these groups are subgroups of the Abelian group $C_q(K)$. We can therefore form the quotient group $H_q(K)$, where $H_q(K) = Z_q(K)/B_q(K)$. The group $H_q(K)$ is referred to as the *qth homology group* of the simplicial complex K. Note that $H_q(K) = 0$ if $q < 0$

or $q > \dim K$ (since $Z_q(K) = 0$ and $B_q(K) = 0$ in these cases). It can be shown that the homology groups of a simplicial complex are topological invariants of the polyhedron of that complex.

The element $[z] \in H_q(K)$ of the homology group $H_q(K)$ determined by $z \in Z_q(K)$ is referred to as the *homology class* of the q-cycle z. Note that $[z_1 + z_2] = [z_1] + [z_2]$ for all $z_1, z_2 \in Z_q(K)$, and $[z_1] = [z_2]$ if and only if $z_1 - z_2 = \partial_{q+1} c$ for some $(q+1)$-chain c.

Proposition 6.4 *Let K be a simplicial complex. Suppose that there exists a vertex $\mathbf{w}$ of K with the following property:*

- *if vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K then so do $\mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$.*

Then $H_0(K) \cong \mathbb{Z}$, and $H_q(K)$ is the zero group for all $q > 0$.

Proof Using Lemma 6.2, we see that there is a well-defined homomorphism $D_q \colon C_q(K) \to C_{q+1}(K)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Now $\partial_1(D_0(\mathbf{v})) = \mathbf{v} - \mathbf{w}$ for all vertices $\mathbf{v}$ of K. It follows that

$$\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle - \left(\sum_{r=1}^{s} n_r \right) \langle \mathbf{w} \rangle = \sum_{r=1}^{s} n_r (\langle \mathbf{v}_r \rangle - \langle \mathbf{w} \rangle) \in B_0(K)$$

for all $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$. But $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition), and thus $H_0(K) = C_0(K)/B_0(K)$. It follows that there is a well-defined surjective homomorphism from $H_0(K)$ to $\mathbb{Z}$ induced by the homomorphism from $C_0(K)$ to $\mathbb{Z}$ that sends $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$ to $\sum_{r=1}^{s} n_r$. Moreover this induced homomorphism is an isomorphism from $H_0(K)$ to $\mathbb{Z}$.

Now let $q > 0$. Then

$$\begin{aligned} &\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle)) \\ &\qquad = \partial_{q+1}(\langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) \\ &\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle \mathbf{w}, \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\ &\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle - D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle)) \end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Thus

$$\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$$

for all $c \in C_q(K)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K)$, and hence $Z_q(K) = B_q(K)$. It follows that $H_q(K)$ is the zero group for all $q > 0$, as required. ∎

Example The hypotheses of the proposition are satisfied for the complex K_σ consisting of a simplex σ together with all of its faces: we can choose $\mathbf{w}$ to be any vertex of the simplex σ.

6.4 Simplicial Maps and Induced Homomorphisms

Any simplicial map $\varphi\colon K \to L$ between simplicial complexes K and L induces well-defined homomorphisms $\varphi_q\colon C_q(K) \to C_q(L)$ of chain groups, where

$$\varphi_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q) \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. (The existence of these induced homomorphisms follows from a straightforward application of Lemma 6.2.) Note that $\varphi_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = 0$ unless $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ are all distinct.

Now $\varphi_{q-1} \circ \partial_q = \partial_q \circ \varphi_q$ for each integer q. Therefore $\varphi_q(Z_q(K)) \subset Z_q(L)$ and $\varphi_q(B_q(K)) \subset B_q(L)$ for all integers q. It follows that any simplicial map $\varphi\colon K \to L$ induces well-defined homomorphisms $\varphi_*\colon H_q(K) \to H_q(L)$ of homology groups, where $\varphi_*([z]) = [\varphi_q(z)]$ for all q-cycles $z \in Z_q(K)$. It is a trivial exercise to verify that if K, L and M are simplicial complexes and if $\varphi\colon K \to L$ and $\psi\colon L \to M$ are simplicial maps then the induced homomorphisms of homology groups satisfy $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

6.5 Connectedness and $H_0(K)$

Lemma 6.5 *Let K be a simplicial complex. Then K can be partitioned into pairwise disjoint subcomplexes $K_1, K_2, \ldots, K_r$ whose polyhedra are the connected components of the polyhedron $|K|$ of K.*

Proof Let $X_1, X_2, \ldots, X_r$ be the connected components of the polyhedron of K, and, for each j, let K_j be the collection of all simplices σ of K for which $\sigma \subset X_j$. If a simplex belongs to K_j for all j then so do all its faces. Therefore $K_1, K_2, \ldots, K_r$ are subcomplexes of K. These subcomplexes are pairwise disjoint since the connected components $X_1, X_2, \ldots, X_r$ of $|K|$ are pairwise disjoint. Moreover, if $\sigma \in K$ then $\sigma \subset X_j$ for some j, since σ is a connected subset of $|K|$, and any connected subset of a topological space is contained in some connected component. But then $\sigma \in K_j$. It follows that $K = K_1 \cup K_2 \cup \cdots \cup K_r$ and $|K| = |K_1| \cup |K_2| \cup \cdots \cup |K_r|$, as required. ∎

The *direct sum* $A_1 \oplus A_2 \oplus \cdots \oplus A_r$ of additive Abelian groups $A_1, A_2, \ldots, A_r$ is defined to be the additive group consisting of all r-tuples $(a_1, a_2, \ldots, a_r)$ with $a_i \in A_i$ for $i = 1, 2, \ldots, r$, where

$$(a_1, a_2, \ldots, a_r) + (b_1, b_2, \ldots, b_r) \equiv (a_1 + b_1, a_2 + b_2, \ldots, a_r + b_r).$$

Lemma 6.6 *Let K be a simplicial complex. Suppose that $K = K_1 \cup K_2 \cup \cdots \cup K_r$, where $K_1, K_2, \ldots K_r$ are pairwise disjoint. Then*

$$H_q(K) \cong H_q(K_1) \oplus H_q(K_2) \oplus \cdots \oplus H_q(K_r)$$

for all integers q.

Proof We may restrict our attention to the case when $0 \leq q \leq \dim K$, since $H_q(K) = \{0\}$ if $q < 0$ or $q > \dim K$. Now any q-chain c of K can be expressed uniquely as a sum of the form $c = c_1 + c_2 + \cdots + c_r$, where c_j is a q-chain of K_j for $j = 1, 2, \ldots, r$. It follows that

$$C_q(K) \cong C_q(K_1) \oplus C_q(K_2) \oplus \cdots \oplus C_q(K_r).$$

Now let z be a q-cycle of K (i.e., $z \in C_q(K)$ satisfies $\partial_q(z) = 0$). We can express z uniquely in the form $z = z_1 + z_2 + \cdots + z_r$, where z_j is a q-chain of K_j for $j = 1, 2, \ldots, r$. Now

$$0 = \partial_q(z) = \partial_q(z_1) + \partial_q(z_2) + \cdots + \partial_q(z_r),$$

and $\partial_q(z_j)$ is a $(q-1)$-chain of K_j for $j = 1, 2, \ldots, r$. It follows that $\partial_q(z_j) = 0$ for $j = 1, 2, \ldots, r$. Hence each z_j is a q-cycle of K_j, and thus

$$Z_q(K) \cong Z_q(K_1) \oplus Z_q(K_2) \oplus \cdots \oplus Z_q(K_r).$$

Now let b be a q-boundary of K. Then $b = \partial_{q+1}(c)$ for some $(q+1)$-chain c of K. Moreover $c = c_1 + c_2 + \cdots c_r$, where $c_j \in C_{q+1}(K_j)$. Thus $b = b_1 + b_2 + \cdots b_r$, where $b_j \in B_q(K_j)$ is given by $b_j = \partial_{q+1} c_j$ for $j = 1, 2, \ldots, r$. We deduce that

$$B_q(K) \cong B_q(K_1) \oplus B_q(K_2) \oplus \cdots \oplus B_q(K_r).$$

It follows from these observations that there is a well-defined isomorphism

$$\nu: H_q(K_1) \oplus H_q(K_2) \oplus \cdots \oplus H_q(K_r) \to H_q(K)$$

which maps $([z_1], [z_2], \ldots, [z_r])$ to $[z_1 + z_2 + \cdots + z_r]$, where $[z_j]$ denotes the homology class of a q-cycle z_j of K_j for $j = 1, 2, \ldots, r$. ∎

Let K be a simplicial complex, and let $\mathbf{y}$ and $\mathbf{z}$ be vertices of K. We say that $\mathbf{y}$ and $\mathbf{z}$ can be joined by an *edge path* if there exists a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ of vertices of K with $\mathbf{v}_0 = \mathbf{y}$ and $\mathbf{v}_m = \mathbf{z}$ such that the line segment with endpoints $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ is an edge belonging to K for $j = 1, 2, \ldots, m$.

Lemma 6.7 *The polyhedron $|K|$ of a simplicial complex K is a connected topological space if and only if any two vertices of K can be joined by an edge path.*

Proof It is easy to verify that if any two vertices of K can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of K. It suffices to verify that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path.

Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path, as required. ∎

Theorem 6.8 *Let K be a simplicial complex. Suppose that the polyhedron $|K|$ of K is connected. Then $H_0(K) \cong \mathbb{Z}$.*

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ be the vertices of the simplicial complex K. Every 0-chain of K can be expressed uniquely as a formal sum of the form

$$n_1\langle\mathbf{u}_1\rangle + n_2\langle\mathbf{u}_2\rangle + \cdots + n_r\langle\mathbf{u}_r\rangle$$

for some integers $n_1, n_2, \ldots, n_r$. It follows that there is a well-defined homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ defined by

$$\varepsilon\left(n_1\langle\mathbf{u}_1\rangle + n_2\langle\mathbf{u}_2\rangle + \cdots + n_r\langle\mathbf{u}_r\rangle\right) = n_1 + n_2 + \cdots + n_r.$$

Now $\varepsilon(\partial_1(\langle \mathbf{y}, \mathbf{z}\rangle)) = \varepsilon(\langle \mathbf{z}\rangle - \langle \mathbf{y}\rangle) = 0$ whenever $\mathbf{y}$ and $\mathbf{z}$ are endpoints of an edge of K. It follows that $\varepsilon \circ \partial_1 = 0$, and hence $B_0(K) \subset \ker \varepsilon$.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ be vertices of K determining an edge path. Then

$$\langle \mathbf{v}_m\rangle - \langle \mathbf{v}_0\rangle = \partial_1 \left(\sum_{j=1}^{m} \langle \mathbf{v}_{j-1}, \mathbf{v}_j\rangle \right) \in B_0(K).$$

Now $|K|$ is connected, and therefore any pair of vertices of K can be joined by an edge path (Lemma 6.7). We deduce that $\langle \mathbf{z}\rangle - \langle \mathbf{y}\rangle \in B_0(K)$ for all vertices $\mathbf{y}$ and $\mathbf{z}$ of K. Thus if $c \in \ker \varepsilon$, where $c = \sum_{j=1}^{r} n_j \langle \mathbf{u}_j\rangle$, then $\sum_{j=1}^{r} n_j = 0$, and hence $c = \sum_{j=2}^{r} n_j(\langle \mathbf{u}_j\rangle - \langle \mathbf{u}_1\rangle)$. But $\langle \mathbf{u}_j\rangle - \langle \mathbf{u}_1\rangle \in B_0(K)$. It follows that $c \in B_0(K)$. We conclude that $\ker \varepsilon \subset B_0(K)$, and hence $\ker \varepsilon = B_0(K)$.

Now the homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ is surjective and its kernel is $B_0(K)$. Therefore it induces an isomorphism from $C_0(K)/B_0(K)$ to $\mathbb{Z}$. However $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition). Thus $H_0(K) \equiv C_0(K)/B_0(K) \cong \mathbb{Z}$, as required. ∎

On combining Theorem 6.8 with Lemmas 6.5 and 6.6 we obtain immediately the following result.

Corollary 6.9 *Let K be a simplicial complex. Then*

$$H_0(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \qquad (r \text{ times}),$$

where r is the number of connected components of $|K|$.