MATH 215B HOMEWORK 4 SOLUTIONS

1. (8 marks) Compute the homology groups of the space X obtained from Δ^n by identifying all faces of the same dimension in the following way: $[v_0, \dots, \hat{v}_j, \dots, v_n]$ is identified with $[v_0, \dots, \hat{v}_k, \dots, v_n]$ by sending each vertex to the vertex in the other simplex that occupies the same place in the ordering $\{v_0, \dots, \hat{v}_k, \dots, v_n\}$ and then extending this map by linearity.

Solution

In order to obtain X, the simplices of Δ^n are identified in such a way to preserve the ordering of the vertices in each face; thus X is a Δ-complex, so we compute the simplicial homology. Since X has at most one simplex of each dimension, we have

$$C_k(X) = \begin{cases} \mathbb{Z} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

The formula for the boundary map of a k-simplex σ is

$$\partial_k \sigma = \sum_{i=0}^{k} (-1)^i \sigma | [v_0, \dots, \hat{v}_i, \dots, v_k]$$

Since we identify all simplices of dimension $k-1$, the boundary map in our chain complex is

$$\partial_k : \mathbb{Z} \xrightarrow{\sum_{i=0}^{k} (-1)^i} \mathbb{Z}$$

so the homomorphism is the identity for k even and the 0 map for k odd. The right end of the chain complex looks like

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

If n is even, the left end of the chain complex looks like

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \cdots$$

and if n is odd, the left end looks like

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \cdots$$

So the cycles are

$$Z_k(X) \approx \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k \text{ odd and } k \le n \\ 0 & \text{otherwise} \end{cases}$$

The boundaries are

$$B_k(X) \approx \begin{cases} \mathbb{Z} & k \text{ odd and } k \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

And the homology is

$$H_k(X) \approx \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

2. (12 marks) (a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown.

Solution

(a) We use the long exact sequence of a pair. In both cases, the homology for X and A vanish above dimension 2, and the exact sequence looks like

$$\cdots \to 0 \to H_n(X, A) \to 0 \to \cdots$$

for $n > 2$, and so the relative homology also vanishes there.

In both cases we have $H_2(A) = H_1(A) = 0$ and $H_2(X) \approx \mathbb{Z}$, so the long exact sequence looks like

$$\cdots \to 0 \to \mathbb{Z} \to H_2(X, A) \to 0 \to \cdots$$

and so $H_2(X, A)$ must be isomorphic to $\mathbb{Z}$.

In both cases, $H_0(X) \approx \mathbb{Z}$ and $H_0(A) \approx \oplus_n \mathbb{Z}$, the direct sum of n copies of $\mathbb{Z}$, where n is the number of points in A. These are free abelian groups generated by the path-components of the respective spaces. The map $H_0(A) \to H_0(X)$ induced by inclusion is the one that sends each generator of $\oplus_n \mathbb{Z}$ to the generator of $\mathbb{Z}$. If $n > 0$, this map is surjective, so by the long exact sequence we have

$$\oplus_n \mathbb{Z} \to \mathbb{Z} \to H_0(X, A) \to 0$$

and $H_0(X, A) = 0$. Of course, if $n = 0$, then $H_k(X, A) = H_k(X, \emptyset) = H_k(X)$.

In the case $X = S^2$, we have $H_1(X) = 0$, and the long exact sequence is

$$\cdots \to 0 \to H_1(X, A) \to \oplus_n \mathbb{Z} \to \mathbb{Z} \to \cdots$$

and so $H_1(X, A) \approx \text{kernel}(\oplus_n \mathbb{Z} \to \mathbb{Z}) \approx \oplus_{n-1} \mathbb{Z}$ (assuming $n > 0$).

In the case $X = S^1 \times S^1$, we have $H_1(X) \approx \mathbb{Z} \oplus \mathbb{Z}$ and $H_1(A) = 0$. So the long exact sequence looks like

$$\cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(X, A) \to \oplus_n \mathbb{Z} \to \mathbb{Z} \to \cdots$$

and so there is a short exact sequence

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(X, A) \to \text{kernel}(\oplus_n \mathbb{Z} \to \mathbb{Z}) \to 0$$

or

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(X, A) \to \oplus_{n-1} \mathbb{Z} \to 0$$

(assuming $n > 0$). Since the last nonzero term is free, the short exact sequence splits, and so $H_1(X, A)$ is isomorphic to the direct sum of the other two terms: $\oplus_{n+1}\mathbb{Z}$. Of course, if $n = 0$, then $H_k(X, A) = H_k(X)$.

Thus, in the case $X = S^2$, we have

$$H_k(X, A) \approx \begin{cases} \oplus_{n-1}\mathbb{Z} & k = 1 \text{ and } n > 0 \\ \mathbb{Z} & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

and in the case $X = S^1 \times S^1$,

$$H_k(X, A) \approx \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & k = 1 \text{ and } n = 0 \\ \oplus_{n+1}\mathbb{Z} & k = 1 \text{ and } n > 0 \\ \mathbb{Z} & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) X/A is homeomorphic to the wedge sum of two tori. So $H_k(X, A) \approx \widetilde{H}_k(X/A) \approx \widetilde{H}_k(S^1 \times S^1) \oplus \widetilde{H}_k(S^1 \times S^1)$. Therefore

$$H_k(X, A) \approx \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & k = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

X/B is homeomorphic to $(S^1 \times S^1)/C$, where $C \subset S^1 \times S^1$ is a set consisting of two points. So by part (a),

$$H_k(X, B) \approx \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

3. (10 marks) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Solution

By example 2.36, we know

$$H_k(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^2 & \text{if } k = 1 \\ \mathbb{Z} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

And the covering space of $S^1 \times S^1$ is $\mathbb{R}^2$, which is contractible so:

$$H_k(\mathbb{R}^2) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, since $S^1 \vee S^1 \vee S^2$ is a CW-complex, by corollary 2.25 we have $\tilde{H}_k(S^1 \vee S^1 \vee S^2) \cong \tilde{H}_k(S^1) \oplus \tilde{H}_k(S^1) \oplus \tilde{H}_k(S^2)$. Using that $S^1 \vee S^1 \vee S^2$ is path-connected, and the homology groups of the spheres, we obtain:

$$H_k(S^1 \vee S^1 \vee S^2) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 1 \\ \mathbb{Z} & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

which is clearly isomorphic to $H_k(S^1 \times S^1)$.

Now, let $p : Y \to S^1 \vee S^1 \vee S^2$ be the universal covering space. We will show that $H_2(Y) \neq 0$. Consider the inclusion $i : S^2 \to S^1 \vee S^1 \vee S^2$. Since $\pi_1(S^2) = 0$, we have $i_*\pi_1(S^2) = 0 = p_*\pi_1(Y)$. By the lifting criterion, there is $f : S^2 \to Y$ such that $pf = i$.

Again by corollary 2.25, the isomorphism $\tilde{H}_k(S^1 \vee S^1 \vee S^2) \cong \tilde{H}_k(S^1) \oplus \tilde{H}_k(S^1) \oplus \tilde{H}_k(S^2)$ is induced by the inclusions, in particular if j and j' are inclusions of the two S^1 summands, then in degree two $j_* \oplus j'_* \oplus i_* : H_2(S^1) \oplus H_2(S^1) \oplus H_2(S^2) \to H_2(S^1 \vee S^1 \vee S^2)$ is an isomorphism. And since $H_2(S^1) = 0$, this says that $i_* : H_2(S^2) \to H_2(S^1 \vee S^1 \vee S^2)$ is an isomorphism. If $H_2(Y) = 0$, since $i_* = p_* f_*$, this would mean that an isomorphism factors through the zero map, which is impossible. Therefore $H_2(Y) \neq 0$ and so the homology of Y is not isomorphic to the homology of $\mathbb{R}^2$.

4. (8 marks) Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with either $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.

Solution

If $f(x) \neq x$ for all $x \in S^n$, then by property (g) in page 134, we would have $\deg(f) = (-1)^{n+1} \neq 0$. On the other hand if $f(y) \neq -y$ for all $y \in S^n$, the line from $f(y)$ to y does not pass through the origin and so we can consider the homotopy

$$H(y, t) = \frac{(1-t)f(y) + ty}{||(1-t)f(y) + ty||}$$

that has $H_0 = f$ and $H_1 = 1$, from where $\deg(f) = \deg(1) = 1 \neq 0$. Therefore both f and $-f$ must have fixed points.

Let $F : D^n \to \mathbb{R}^n$ be a vector field such that $F(x) \neq 0$ for all x. Then consider the map $a : D^n \to S^{n-1}$ defined by $a(z) = F(z)/||F(z)||$ and the restriction to the boundary $b : \partial D^n = S^{n-1} \to S^{n-1}$. The map b factors through D^n, which is contractible, therefore it must be nullhomotopic. In particular, it has degree 0. By the first part of the problem, there are points x and y in ∂D^n with $b(x) = x$ and $b(y) = -y$. But this means $F(x) = ||F(x)||x$, which is a vector pointing radially outward and $F(y) = ||F(y)||(-y)$, which is a vector pointing radially inward.

5. (12 marks) Compute the homology groups of the following 2-complexes:

(a) The quotient of S^2 obtained by identifying north and south poles to a point.

(c) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.

Solution

(a) The north and south poles comprise a space homeomorphic to S^0, so the quotient space of interest can be denoted S^2/S^0. We use the long exact sequence of the pair (S^2, S^0). The bit involving $H_2(S^2, S^0)$ looks like:

$$0 \to H_2(S^2) \to H_2(S^2, S^0) \to 0$$

so $H_2(S^2, S^0) \approx H_2(S^2) \approx \mathbb{Z}$. The bit involving $H_1(S^2, S^0)$ looks like:

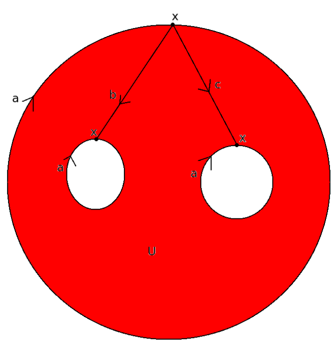
$$0 \to H_1(S^2, S^0) \to H_0(S^0) \to H_0(S^2)$$

so $H_1(S^2, S^0) \approx \text{kernel}(H_0(S^0) \to H_0(S^2)) \approx \mathbb{Z}$.

S^2/S^0 has a 2-dimensional cell structure, so it has trivial homology in higher dimensions. (This also follows from the long exact sequence.) Now we know $H_n(S^2/S^0) \approx \widetilde{H}_n(S^2, S^0) = H_n(S^2, S^0)$ for $n > 0$. The 0-dimensional homology is $\mathbb{Z}$ because S^2/S^0 is path-connected.

$$H_n(S^2/S^0) \approx \begin{cases} \mathbb{Z} & \text{for } n \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(c) This is a CW complex X with one 0-cell x, three 1-cells a, b, c and one 2-cell U, as in the following picture.



So the cellular complex is given by $C_2^{CW}(X) = \mathbb{Z} < U >$, $C_1^{CW}(X) = \mathbb{Z} < a, b, c >$, and $C_0^{CW} = \mathbb{Z} < x >$.

Note that the 2-cell is attached to the loop $aba^{-1}b^{-1}ca^{-1}c^{-1}$, so $d_2(U) = -a$. There is only one 0-cell, so $d_1 = 0$.

Since X is a 2-dimensional CW-complex, $H_n(X) = 0$ for $n \geq 3$. The map d_2 is injective, so $H_2(X) = 0$. And $\text{Ker}(d_1)/\text{Im}(d_2) = \mathbb{Z} < a, b, c > /\mathbb{Z} < a > \cong \mathbb{Z}^2$. Finally, X is path connected, so $H_0(X) = \mathbb{Z}$.

Therefore $H_n(X) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^2 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$

6. (10 marks) Show that $H_i(X \times S^n) \approx H_i(X) \oplus H_{i-n}(X)$ for all i and n, where $H_i = 0$ for $i < 0$ by definition. Namely, show $H_i(X \times S^n) \approx H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$ and $H_i(X \times S^n, X \times \{x_0\}) \approx H_{i-1}(X \times S^{n-1}, X \times \{x_0\})$. [For the latter isomorphism the relative Meyer-Vietoris sequence yields an easy proof.]

Solution

There is a retraction $r : X \times S^n \to X \times \{x_0\}$ given by $r(x, s) = (x, x_0)$. This means, first of all, that the boundary map in the long exact sequence of the pair is trivial. Indeed, if we let i denote the inclusion, we have $ri = \mathbb{1}_{X\times\{x_0\}}$, and so $r_* i_* = \mathbb{1}_{H_k(X\times\{x_0\})}$; in particular, i_* is injective, so the map preceding it in the long exact sequence is trivial.

So the long exact sequence breaks up into short exact sequences

$$0 \to H_k(X \times \{x_0\}) \to H_k(X \times S^n) \to H_k(X \times S^n, X \times \{x_0\}) \to 0$$

Furthermore, r is a splitting of this short exact sequence, so the middle term is isomorphic to $H_k(X \times \{x_0\}) \oplus H_k(X \times S^n, X \times \{x_0\}) \approx H_k(X) \oplus H_k(X \times S^n, X \times \{x_0\})$.

For the next step, we use Mayer-Vietoris. For any m, the sphere S^m is the union of two closed hemispheres, which are homeomorphic to disks. If we make the disks slightly larger we get a covering of S^m by two open disks D^m_+ and D^m_- which are contractible and whose intersection deformation-retracts onto the equator S^{m-1}. We can ensure that the basepoint x_0 lies in $D^m_+ \cap D^m_-$. Now let $A = X \times D^m_+$, $B = X \times D^m_-$, and $C = D = X \times \{x_0\}$, so that

- $(A, C) \simeq (X, X)$
- $(B, D) \simeq (X, X)$
- $(A \cup B, C \cup D) = (X \times S^m, X \times \{x_0\})$
- $(A \cap B, C \cap D) \simeq (X \times S^{m-1}, X \times \{x_0\})$

In the Mayer-Vietoris sequence for this covering, the term $H_k(A, C) \oplus H_k(B, D) \approx H_k(X, X) \oplus H_k(X, X)$ vanishes, so we have

$$0 \to H_k(A \cup B, C \cup D) \to H_{k-1}(A \cap B, C \cap D) \to 0$$

This is true even if $k = 0$. So

$$H_k(A \cup B, C \cup D) \approx H_{k-1}(A \cap B, C \cap D)$$
$$H_k(X \times S^m, X \times \{x_0\}) \approx H_{k-1}(X \times S^{m-1}, X \times \{x_0\})$$

Now iterating this last isomorphism, we find that $H_k(X \times S^n, X \times \{x_0\}) \approx H_{k-n}(X \times S^0, X \times \{x_0\}) \approx H_{k-n}(X \coprod X, X) \approx H_{k-n}(X)$. (The last isomorphism is excision.) Combining this with our first result, we have

$$H_k(X \times S^n) \approx H_k(X) \oplus H_{k-n}(X)$$