LECTURE 4: SINGULAR HOMOLOGY OF CONTRACTIBLE SPACES

In the last lecture we introduced relative singular homology of a pair of spaces. Let us quickly recall the construction. Given a space X and a subspace A together with its inclusion $i\colon A \to X$ then we have the induced short exact sequence of singular chain complexes:

$$0 \to C(A) \to C(X) \to C(X,A) = C(X)/C(A) \to 0$$

The quotient complex $C(X,A)$ is the *relative singular chain complex* and its homology is the *relative singular homology of the pair*:

$$H_n(X,A) = H_n(C(X,A))$$

By construction of the quotient complex chains in it can be represented by singular chains c in X. Moreover, such a chain $c \in C_n(X)$ represents a cycle in the relative chain complex if its usual singular boundary $\partial(c)$ lies in the image of $C_{n-1}(A) \to C_{n-1}(X)$. In this situation we say that c is a *cycle mod* A or a *cycle relative to* A. Thus, relative singular homology classes, in general, can not be represented by cycles in X but always by cycles relative to A. We leave it to the reader to define the category Top^2 of pairs of spaces and to remark that relative singular chain complexes and relative singular homology groups define functors $C\colon \mathsf{Top}^2 \to \mathsf{Ch}(\mathbb{Z})$ and $H_n\colon \mathsf{Top}^2 \to \mathsf{Ab}$ respectively (Exercise!).

The aim of this and the next lecture is to show that (relative) singular homology is *homotopy-invariant*. Recall that it is an immediate consequence of the functoriality of singular homology that *homeomorphic* spaces have naturally isomorphic homology groups. We want to show next that this also holds true for *homotopy equivalent* spaces. In fact, this will be a consequence of the more general result that homotopic maps induce the same maps on singular homology.

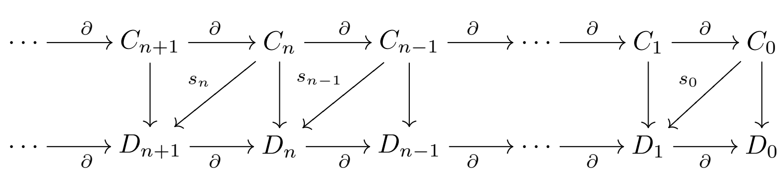
We begin by splitting of an algebraic definition which 'mimics' the notion of a homotopy at the level of chain complexes. With our main application in mind we restrict attention to non-negative chain complexes. Recall that given two chain complexes C, D of abelian groups, then a chain map $f\colon C \to D$ consists of a family of group homomorphisms $f_n\colon C_n \to D_n$ which commute with the differentials.

Definition 1. Let $C, D \in \mathsf{Ch}(\mathbb{Z})$ be chain complexes and let $f, g\colon C \to D$ be chain maps. A *chain homotopy* s from f to g, denoted $s\colon f \simeq g$, consists of group homomorphisms $s_n\colon C_n \to D_{n+1}$ for all $n \geq 0$ such that:

$$\partial \circ s_n + s_{n-1} \circ \partial \quad = \quad g_n - f_n, \quad n \geq 1, \qquad \text{and} \qquad \partial \circ s_0 \quad = \quad g_0 - f_0$$

Two chain maps f and g are *chain homotopic*, denoted $f \simeq g$, if there is such a chain homotopy.

Thus, in the situation of the definition we have the following diagram in which the vertical homomorphisms are given by $g_n - f_n$ for the corresponding value of n:



One reason why we are interested in this concept is given by Lemma 3. But let us first collect some elementary facts about chain homotopies (the precise formulations are left to the reader).

Lemma 2. *The chain homotopy relation defines an equivalence relation on the set of chain maps between two chain complexes. Moreover, it is compatible with composition, addition, and the formation of additive inverses.*

Proof. Exercise. □

Lemma 3. *Let C, $D \in \mathsf{Ch}(\mathbb{Z})$ and let us consider chain maps $f, g\colon C \to D$. If f and g are chain homotopic then the induced maps in homology are equal, i.e., we have:*

$$H_n(f) = H_n(g)\colon \quad H_n(C) \to H_n(D), \qquad n \geq 0$$

Proof. Let $s\colon f \simeq g$ be a chain homotopy and let us consider a homology class $\omega = [z_n] \in H_n(C)$. For $n \geq 1$ we calculate

$$H_n(g)(\omega) \quad = \quad [g_n(z_n)] \quad = \quad [f_n(z_n) + \partial s_n(z_n) + s_{n-1}\partial(z_n)] \quad = \quad [f_n(z_n)] \quad = \quad H_n(f)(\omega)$$

The third equality uses that z_n is a cycle and that a homology class is not changed if we add a boundary. In degree zero it is even slightly simpler since we can conclude the proof by:

$$H_0(g)(\omega) \quad = \quad [g_0(z_0)] \quad = \quad [f_0(z_0) + \partial s_0(z_0)] \quad = \quad [f_0(z_0)] \quad = \quad H_0(f)(\omega)$$ □

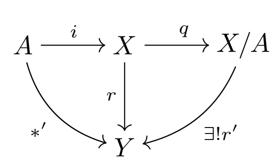
Thus, in order to establish the homotopy invariance of singular homology we have to show that homotopic maps between topological spaces induce chain homotopic maps between the corresponding singular chain complexes. However, this will require some preparation and will only be completed in the next lecture.

Let us begin by recalling the quotient of a space by a subspace. We first construct the underlying set of the quotient space. Any pair of spaces (X, A) with inclusion map $i\colon A \to X$ induces an equivalence relation on X, namely the equivalence relation $\sim_A$ generated by $i(a) \sim_A i(a')$ for all a, $a' \in A$. Let us denote the set of equivalence classes with respect to $\sim_A$ by X/A:

$$X/A \quad = \quad X/\sim_A$$

There is the natural quotient map $q\colon X \to X/A$ which sends an element $x \in X$ to its equivalence class $[x] \in X/A$. The equivalence class of $i(a)$ for an arbitrary $a \in A$ will be denoted by $*$. We endow X/A with the finest topology such that the quotient map $q\colon X \to X/A$ is continuous. In other words, a subset $U \subset X/A$ is open by definition if and only if $q^{-1}(U)$ is open in X. It is immediate from the construction that the composition $q \circ i\colon A \to X \to X/A$ is constant with value $*$. This quotient space construction has the following universal property.

Lemma 4. *Let (X, A) be a pair of spaces and let $(X/A, q)$ be the quotient space X/A together with its quotient map $q\colon X \to X/A$. Given a further such pair (Y, r) consisting of a topological space Y and a continuous map $r\colon X \to Y$ such that the composition $r \circ i\colon A \to Y$ is constant then there is a unique continuous map $r'\colon X/A \to Y$ such that $r' \circ q = r$:*



Proof. Exercise. □

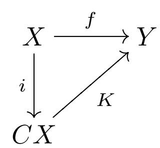
We now apply this to the cone construction of a topological space. Let X be a topological space then the product $I \times X$ where $I = [0,1]$ is the *cylinder of* X. The *cone* CX of X is obtained from the cylinder by collapsing the top:

$$I \times X \quad \stackrel{p}{\to} \quad CX \quad = \quad I \times X/\{1\} \times X$$

Elements of this space are equivalence classes $[t,x]$, $t \in I$, $x \in X$ and we have $[1,x] = [1,x']$ for all x, $x' \in X$. The point given by $[1,x]$ is called the *apex of the cone* and will be denoted by $*$. We obtain an inclusion of X in the cone by composing the inclusion $X \to I \times X : x \mapsto (0,x)$ with the quotient map to CX:

$$i\colon X \to I \times X \stackrel{p}{\to} CX\colon \qquad x \mapsto [0,x]$$

Lemma 5. *Let $f\colon X \to Y$ be a map of topological spaces. Then f is homotopic to a constant map if and only if f extends over the cone CX in the sense that there is a map $K\colon CX \to Y$ such that the following diagram commutes:*



Proof. Given a homotopy $H\colon I \times X \to Y$ with $H_0 = f$ and $H_1 = \kappa_y$ the constant map at $y \in Y$ obviously factors over CX to give the desired map K. Conversely, if we have such a map K then we can precompose it with $p\colon I \times X \to CX$ to obtain a homotopy H between f and a constant map. That these two assignments are inverse bijections is just a special case of the last lemma. □

More precisely, the proof shows that there is a bijection between homotopies to constant maps and extensions over the cone. In the special case of $X = \Delta^n$ we have a homeomorphism $C\Delta^n \cong \Delta^{n+1}$ such that the apex of the cone corresponds to $e_0 \in \Delta^{n+1}$. A precise formula for this homeomorphism reads as follows where we use the short hand notation $t' = t_1 + \ldots + t_{n+1}$ and where t_i are the barycentric coordinates:

$$\phi\colon \Delta^{n+1} \stackrel{\cong}{\to} C\Delta^n\colon \quad (t_0, \ldots, t_{n+1}) \mapsto \begin{cases} [t_0, t_1/t', \ldots, t_{n+1}/t'] & , \quad t' \neq 0 \\ * & , \quad t' = 0 \end{cases}$$

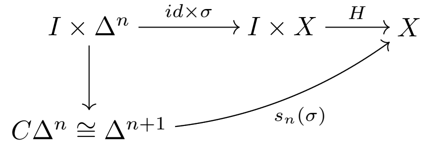
Moreover, this homeomorphism has the nice property that it identifies the face map

$$d^0\colon \Delta^n \to \Delta^{n+1}\colon (t_0, \ldots, t_n) \mapsto (0, t_0, \ldots, t_n)$$

with the inclusion $i\colon \Delta^n \to C\Delta^n$, i.e., we have $\phi \circ d^0 = i\colon \Delta^n \to C\Delta^n$. Thus, we have a bijection between homotopies of $\sigma\colon \Delta^n \to Y$ to constant maps and extensions of σ to maps $s(\sigma)\colon \Delta^{n+1} \to Y$ such that $s(\sigma) \circ d^0 = \sigma$.

Proposition 6. *Let X be a contractible space, then all homology groups $H_n(X)$ vanish for $n \geq 1$.*

Proof. Let X be contractible, i.e., we can find a point $x_0 \in X$ and a homotopy $H\colon I \times X \to X$ with $H_0 = id_X$ and $H_1 = \kappa_{x_0}$, the constant map at x_0. Now, let $\sigma\colon \Delta^n \to X$ be a basis element of the singular chain group $C_n(X)$. Precomposition of H with $id_I \times \sigma$ yields a further such homotopy as indicated in the first row of the following diagram:



Now, the composition $H \circ (id \times \sigma)$ gives us a homotopy to a constant map so that it corresponds to a unique map $s_n(\sigma)\colon \Delta^{n+1} \to X$. Additive extension gives us a group homomorphism:

$$s_n\colon C_n(X) \to C_{n+1}(X)$$

By construction, it follows that we have the following relations for the faces of $s_n(\sigma)$:

$$d_0(s_n\sigma) = \sigma, \quad n \geq 0, \qquad \text{and} \qquad d_i(s_n(\sigma)) = s_{n-1}(d_{i-1}(\sigma)), \quad i = 1, \ldots, n+1, \quad n \geq 1$$

In the remaining case $n = 0$ and $i = 1$ we have $d_1(s_0\sigma) = x_0 \in C_0(X)$, where we use the standard convention that x_0 also denotes the map $\Delta^0 \to X$ sending the unique point in Δ^0 to x_0. But these relations allow us to make the following calculation for $n \geq 1$:

$$\begin{aligned}
\partial s_n(\sigma) &= \sum_{i=0}^{n+1} (-1)^i d_i(s_n(\sigma)) \\
&= d_0(s_n(\sigma)) + \sum_{i=1}^{n+1} (-1)^i d_i(s_n\sigma) \\
&= \sigma + \sum_{i=1}^{n+1} (-1)^i s_{n-1}(d_{i-1}\sigma) \\
&= \sigma - s_{n-1}(\partial\sigma)
\end{aligned}$$

By linear extension we thus deduce for $n \geq 1$ the relation $\partial \circ s_n + s_{n-1} \circ \partial = id$. In the remaining degree $n = 0$ where an 0-simplex is just given by a point $x\colon \Delta^0 \to X$ we have:

$$\partial s_0(x) \quad = \quad d_0(s_0(x)) - d_1(s_0(x)) \quad = \quad x - x_0$$

Thus, we have the relation $\partial \circ s_0 = id - \epsilon_0$ where $\epsilon_0\colon C_0(X) \to C_0(X)$ sends each basis element to x_0. As an upshot of these calculations we thus constructed a chain homotopy $s\colon id \to \epsilon$ where ϵ is the chain map (!) which is zero in all positive degrees and ϵ_0 in degree 0. Thus, by Lemma 3 we have $id = 0\colon H_n(X) \to H_n(X)$ for $n \geq 1$ which can only be the case if these groups vanish. □

This proposition implies that the homology of a large family of spaces vanishes in positive dimensions. This applies to points, vector spaces, discs, simplices, products of such spaces etc. In some sense we are happy with this result since the motivational idea of homology was that we want to have invariants which measure the 'geometric complexity' of spaces. And it is good to know that the invariants are trivial in these examples.

Furthermore, this result (applied to products of simplices) is essential in our approach to the homotopy-invariance of singular homology. Given two spaces X, Y we want to relate the singular chain complexes $C(X), C(Y), C(X \times Y)$ and this relation is to be natural in the spaces. Recall that given maps $f\colon X \to X'$ and $g\colon Y \to Y'$ then there is the product map $(f, g)\colon X \times Y \to X' \times Y'$ which sends (x, y) to $(f(x), g(y))$.

Theorem 7. *Given topological spaces X, Y then there are bilinear maps*

$$\times\colon C_p(X) \times C_q(Y) \to C_{p+q}(X \times Y)\colon \quad (c, d) \mapsto c \times d$$

for all p, $q \geq 0$ called cross product maps *with the following properties:*
i) *For $x \in X$, $y \in Y$, $\sigma\colon \Delta^p \to X$, and $\tau\colon \Delta^q \to Y$ we have:*

$$x \times \tau\colon \Delta^q \cong \Delta^0 \times \Delta^q \stackrel{(x,\tau)}{\to} X \times Y \qquad \text{and} \qquad \sigma \times y\colon \Delta^p \cong \Delta^p \times \Delta^0 \stackrel{(\sigma,y)}{\to} X \times Y$$

ii) *The cross product is natural in X and Y, i.e., for maps $f\colon X \to X'$ and $g\colon Y \to Y'$ we have:*

$$(f,g)_*(c \times d) \quad = \quad f_*(c) \times g_*(d) \qquad \textit{in} \qquad C_{p+q}(X' \times Y')$$

iii) *The boundary ∂ is a* derivation *with respect to $\times$ in the sense that for $c \in C_p(X)$ and $d \in C_q(Y)$ we have:*

$$\partial(c \times d) \quad = \quad \partial(c) \times d \; + \; (-1)^p c \times \partial(d) \qquad \textit{in} \qquad C_{p+q-1}(X \times Y)$$

We will begin the next lecture with a proof of this theorem. A crucial step in this proof will use that spaces of the form $\Delta^p \times \Delta^q$ have trivial homology groups in positive dimensions.