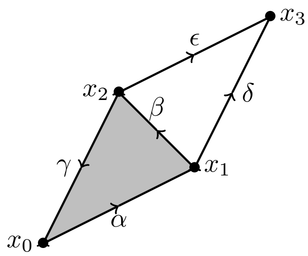
LECTURE 1: DEFINITION OF SINGULAR HOMOLOGY

As a motivation for the notion of *homology* let us consider the topological space X which is obtained by gluing a solid triangle to a 'non-solid' triangle as indicated in the following picture. The vertices and some paths (with orientations) are named as indicated in the graphic.



Let us agree that we define the boundary of such a path by the formal difference 'target - source'. So, the boundary $\partial(\beta)$ of β is given by $\partial(\beta) = x_2 - x_1$. In this terminology, the geometric property that a path is closed translates into the algebraic relation that its boundary vanishes. Moreover, let us define a *chain of paths* to be a formal sum of paths. In our example, we have the chains $c_1 = \alpha + \beta + \gamma$ and $c_2 = \beta + \epsilon + \delta^{-1}$. Both c_1 and c_2 are examples of closed paths (this translates into the algebraic fact that the sum of the boundaries of the paths vanishes). However, from a geometrical perspective, both chains behave very differently: c_1 is the boundary of a solid triangle (and is hence closed for trivial reasons) while c_2 is not of that form. Thus, the chain c_2 detects some 'interesting geometry'.

The basic idea of homology is to systematically measure closed chains of paths (which might be interesting) and divide out by the 'geometrically boring ones'. Moreover, we would like to extend this to higher dimensions. Let us now begin with a precise development of the theory.

Definition 1. Let $n \geq 0$ be a natural number. The (**geometric**) n**-simplex** $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the convex hull of the standard basis vectors of $\mathbb{R}^{n+1}$ endowed with the subspace topology.

Let us denote these standard basis vectors by $e_0, \ldots, e_n$. Every point $v \in \Delta^n$ can *uniquely* be written as a convex linear combination of the e_i, i.e., there is a unique expression

$$v = \Sigma_{i=0}^n t_i e_i, \qquad t_i \geq 0, \qquad t_0 + \ldots + t_n = 1.$$

The coordinates t_i are the **barycentric coordinates** of the point v. Thus, to be completely specific, we have

$$\Delta^n = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \quad t_i \geq 0, \quad t_0 + \ldots + t_n = 1\}.$$

Recall that a convex linear map is a map which sends convex linear combinations to convex linear combinations. It follows that a convex linear map

$$\alpha \colon \Delta^n \to \Delta^m$$

is uniquely determined by its values on $e_i \in \Delta^n$ for $i = 0, \ldots, n$. In the important case that α sends vertices to vertices, i.e., if we have $\alpha(e_i) = e_{a(i)}$ for a certain map of sets $a \colon \{0, \ldots, n\} \to \{0, \ldots, m\}$, then we obtain

$$\alpha(\Sigma_{i=0}^n t_i e_i) \quad = \quad \Sigma_{i=0}^n t_i e_{a(i)} \quad = \quad \Sigma_{j=0}^m s_j e_j \qquad \text{with} \qquad s_j = \Sigma_{a(i)=j} t_i.$$

As a special case we have the **face maps**

$$d^i\colon \Delta^{n-1} \to \Delta^n, \quad (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n), \qquad 0 \leq i \leq n.$$

This map is determined by the unique injective, monotone map $\{0, \ldots, n-1\} \to \{0, \ldots, n\}$ which does not hit i. The image $d^i(\Delta^{n-1}) \subseteq \Delta^n$ is called the **i-th face** of Δ^n. For example, the possible face maps $\Delta^0 \to \Delta^1$ are the inclusions of the 'target' or the 'source', namely

$$t = d^0\colon \Delta^0 \to \Delta^1 \qquad \text{and} \qquad s = d^1\colon \Delta^0 \to \Delta^1.$$

In the next dimension, we have three face maps

$$d^i\colon \Delta^1 \to \Delta^2,\ i = 0, \ldots, 2$$

given by the inclusions of the three 'sides' of the topological boundary of $\Delta^2 \subseteq \mathbb{R}^3$. We strongly recommend the reader to draw the corresponding pictures.

For iterated face maps there is the following key relation (it is a special case of the **cosimplicial identities**) which 'lies at the heart of many kinds of homology theories'.

Lemma 2. *For every $n \geq 2$ and every $0 \leq j < i \leq n$ the following iterated face maps coincide*

$$d^i \circ d^j = d^j \circ d^{i-1}\colon \Delta^{n-2} \to \Delta^n.$$

Proof. It is immediate to verify that both maps are given by

$$(t_0, t_1, \ldots, t_{n-2}) \mapsto (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-2}).$$

In fact, both iterated face maps are determined by the unique monotone injection

$$\{0, \ldots, n-2\} \to \{0, \ldots, n\}$$

which hits neither i nor j. $\square$

The idea of singular homology consists of studying an arbitrary space by considering formal sums of maps defined on simplices of a fixed dimension.

Definition 3. Let X be a topological space.

(1) A **singular n-simplex in** X is a continuous map $\sigma\colon \Delta^n \to X$.
(2) The **singular n-chain group** $C_n(X)$ is the free abelian group generated by the singular n-simplices in X. Its elements are called **singular n-chains in** X.

Let us recall the notion of a free abelian group generated by a set. As a motivation for the concept we include the following reminder.

Reminder 4. Let V be a finite-dimensional vector space with basis $b_1, \ldots, b_n \in V$ and let W be a further vector space over the same field. Then a linear map $f\colon V \to W$ is uniquely determined by the values $f(b_1), \ldots, f(b_n) \in W$.

Definition 5. Let S be a set. A **free abelian group generated by** S is a pair $(F(S), i_S)$ consisting of an abelian group $F(S)$ and a map of sets $i_S\colon S \to F(S)$ which satisfies the following universal property: Given a further pair $(A, j\colon S \to A)$ with A an abelian group and j a map of sets then there is a unique group homomorphism $f\colon F(S) \to A$ such that $f \circ i_S = j$.

More diagrammatically, this universal property can be depicted as follows (the reason why we split the diagram into two parts will become more apparent later: the left part is a diagram of sets, while the right part is a diagram of abelian groups):

$$\begin{tikzcd} S \arrow[r, "i_S"] \arrow[dr, bend right, "\forall j"'] \arrow[dr, phantom, "="] & F(S) \arrow[d, dashed, "f"] \\ & A \end{tikzcd} \qquad\qquad \begin{tikzcd} F(S) \arrow[d, "\exists! f"] \\ A \end{tikzcd}$$

The next lemma establishes the existence and essential uniqueness of free abelian groups generated by a set. This motivates us to think of the free abelian group generated by a set as 'the' best approximation of a set by an abelian group.

Lemma 6. (1) *Let S be a set. The free abelian group $(F(S), i_S)$ exists and is unique up to a unique isomorphism compatible with the maps from S. In more detail, given a further free abelian group $(F(S)', i'_S)$ then there is a unique isomorphism of groups $\phi\colon F(S) \to F(S)'$ which satisfies the relation $\phi \circ i_S = i'_S$:*

$$\begin{tikzcd} S \arrow[r, "i_S"] \arrow[dr, bend right, "i'_S"'] \arrow[dr, phantom, "="] & F(S) \arrow[d, "\phi"] \\ & F(S)' \end{tikzcd} \qquad\qquad \begin{tikzcd} F(S) \arrow[d, "\exists!\, \phi"] \\ F(S)' \end{tikzcd}$$

(2) *The assignment $S \mapsto F(S)$ sending a set S to a free abelian group generated by S can be extended to a* free abelian group functor *from the category* Set *of sets to the category* Ab *of abelian groups:*

$$F\colon \mathsf{Set} \to \mathsf{Ab}$$

Proof. Exercise. □

The proof of this lemma will show that every element of $F(S)$ can be written as a finite sum of elements in S, i.e, for $z \in F(S)$ we have

$$z = n_1 s_1 + \ldots + n_k s_k, \quad n_i \in \mathbb{Z}, \quad s_i \in S, \quad i = 1, \ldots, k.$$

Moreover, this expression is unique up to a permutation of the summands if we insist that the n_i are different from 0 and that the s_i are pairwise different. In particular, this applies to the singular chain group $C_n(X)$ associated to a topological space X. Thus, a singular n-chain can be written as a formal sum of singular n-simplices in X.

Let $f\colon X \to Y$ be a map of spaces (unless stated differently all maps between spaces will be assumed to be continuous). Given a singular n-simplex $\sigma\colon \Delta^n \to X$ then $f \circ \sigma\colon \Delta^n \to Y$ is a singular n-simplex in Y. The linear extension of this assignment (whose existence is guaranteed by the last lemma) defines a group homomorphism:

$$C_n(f) = f_*\colon C_n(X) \to C_n(Y)$$

Corollary 7. *The assignments $X \mapsto C_n(X)$ and $f \mapsto f_*$ define a functor, the* singular n-chain group functor *C_n, from the category* Top *of topological spaces to the category* Ab *of abelian groups:*

$$C_n\colon \mathsf{Top} \to \mathsf{Ab}$$

The next aim is to relate the singular chain group functors of the various dimensions. For this purpose, recall that we have the i-th face map $d^i\colon \Delta^{n-1}\to\Delta^n$ for $0\leq i\leq n$. Given a singular n-simplex $\sigma\colon \Delta^n\to X$ in a space X we obtain a singular $(n-1)$-simplex $d_i(\sigma)$ in X by setting:

$$d_i(\sigma)=\sigma\circ d^i\colon \Delta^{n-1}\stackrel{d^i}{\to}\Delta^n\stackrel{\sigma}{\to}X,\qquad 0\leq i\leq n$$

By linear extension this gives rise to a group homomorphism

$$d_i\colon C_n(X)\to C_{n-1}(X),\quad 0\leq i\leq n,$$

which will also be called the ***i*-th face map**. The key definition of the entire business is the following one.

Definition 8. Let X be a topological space.

(1) The n-th **singular boundary operator** ∂ is given by

$$\partial=\sum_{i=0}^n(-1)^i d_i\colon C_n(X)\to C_{n-1}(X).$$

(2) The kernel $Z_n(X)$ of the boundary operator $\partial\colon C_n(X)\to C_{n-1}(X)$, i.e., the abelian group

$$Z_n(X)=\ker(\partial\colon C_n(X)\to C_{n-1}(X)),$$

is the group of **singular *n*-cycles** in X. An element of $Z_n(X)$ is sometimes also referred to as a **closed** singular n-chain.

(3) The image $B_n(X)$ of the boundary operator $\partial\colon C_{n+1}(X)\to C_n(X)$, i.e., the abelian group

$$B_n(X)=\mathrm{im}(\partial\colon C_n(X)\to C_{n-1}(X)),$$

is the group of **singular *n*-boundaries** in X.

Thus, by forming the alternating sum of the face maps we obtain a map between the groups in various dimensions and this gives rise to two subgroups

$$B_n(X),Z_n(X)\subseteq C_n(X),\quad n\geq 0.$$

In the special case of $n=0$ we define $Z_0(X)=C_0(X)$, i.e., every 0-chain is by definition also a 0-cycle. A key property of these boundary maps is given in the next proposition. Once one gets used to the calculation in its proof, one remarks that the proposition is an immediate consequence of the cosimplicial identity in Lemma 2.

Proposition 9. *Given a topological space X then the singular boundary maps define a **differential** on $\{C_\bullet(X)\}$, i.e., we have the relations*

$$\partial\circ\partial=0\colon C_n(X)\to C_{n-2}(X),\qquad n\geq 2.$$

Proof. This follows from the following algebraic manipulation in which Lemma 2 plays a key role and where the last step is given by a shift of the inner summation index.

$$\begin{aligned}
\partial \circ \partial &= \sum_{j=0}^{n-1}\sum_{i=0}^{n}(-1)^{i+j} d_j \circ d_i \\
&= \sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^{i+j} d_j \circ d_i \quad + \quad \sum_{j=0}^{n-1}\sum_{i=j+1}^{n}(-1)^{i+j} d_j \circ d_i \\
&\overset{!}{=} \sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^{i+j} d_j \circ d_i \quad + \quad \sum_{j=0}^{n-1}\sum_{i=j+1}^{n}(-1)^{i+j} d_{i-1} \circ d_j \\
&= \sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^{i+j} d_j \circ d_i \quad + \quad \sum_{j=0}^{n-1}\sum_{i=j}^{n-1}(-1)^{i+j+1} d_i \circ d_j
\end{aligned}$$

If we now interchange the roles of i and j in the –say– second sum we remark that the sums cancel each other as intended. □

Definition 10. The **singular chain complex** of a topological space X is the pair $(C_*(X), \partial)$ consisting of the singular chain groups together with the singular boundary operators:

$$\ldots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X)$$

We will usually abuse notation and simply write $C(X)$ or $C_*(X)$ for the singular chain complex. The above proposition is very important. It implies that in each dimension n we have inclusions of subgroups

$$B_n(X) \subseteq Z_n(X) \subseteq C_n(X).$$

Moreover, since all groups occurring here are abelian, the subgroups are normal subgroups so that the following definition makes sense.

Definition 11. Let X be a topological space. The n-th **singular homology group** $H_n(X)$ of X is the abelian group defined by

$$H_n(X) = Z_n(X)/B_n(X).$$

This definition completes the program motivated by our initial example. Given a topological space we can associate to it an abelian group which is obtained by taking the singular *cycles* in a fixed dimension (which *might be* geometrically interesting) and by dividing out those which are geometrically uninteresting. The result of this gives us by definition the singular homology of the space in that fixed dimension.

Given two singular n-cycles $z_1,\, z_2 \in Z_n(X)$ which represent the same homology class, i.e., their difference is a boundary, are called **homologous**. This will be denoted by:

$$z_1 \sim z_2 \quad :\Longleftrightarrow \quad z_1 - z_2 \in B_n(X)$$

For example a cycle z is a boundary if and only if $z \sim 0$.

Example 12. (1) Let $*$ denote the space consisting of one point only. Then we have $H_0(*) \cong \mathbb{Z}$ and $H_n(*) \cong 0$ for $n \geq 1$. (Exercise.)

(2) The formation of singular homology is *additive* in the following sense. Let X be a topological space and let $X_\alpha,\ \alpha \in I$, be its path components, then there is a (natural) isomorphism

$$\bigoplus_{\alpha\in I} H_n(X_\alpha) \overset{\cong}{\to} H_n(X), \qquad n \geq 0.$$

Thus, for many calculations it suffices to restrict attention to path-connected spaces. However, certain statements might be nicer if we allow for more general spaces (see for example the next proposition). We will prove the claimed additivity in a later lecture. Although one could already easily make precise the definition of the above map we prefer to first establish the functoriality of singular homology. Of course, the reader is invited to convince her- or himself that such a relation should be true.

We close this lecture by the following low-dimensional identification. Associated to a connected space X we have the following (natural) **augmentation map** ϵ. Recall that $C_0(X)$ is freely generated by the singular 0-simplices in X. Thus, an element of this group is just a formal sum of points in X. Sending each point of X to $1 \in \mathbb{Z}$ and then forming the linear extension gives rise to the augmentation map

$$\epsilon\colon C_0(X) \to \mathbb{Z}\colon \quad \sum_{i=1}^{k} n_i x_i \mapsto \sum_{i=1}^{k} n_i.$$

By our convention in dimension 0 we have $C_0(X) = Z_0(X)$. Now, it is easy to check that the augmentation map vanishes on all 0-boundaries. Thus, the universal property of the quotient of abelian groups (see the exercises) implies that we get a unique induced group homomorphism ϵ_* as indicated in:

$$\begin{tikzcd}
B_0(X) \arrow[r, "i"] \arrow[dr, bend right, "0"'] & Z_0(X) \arrow[r, "q"] \arrow[d, "\epsilon"] & H_0(X) \arrow[dl, bend left, "\epsilon_*"] \\
& \mathbb{Z} &
\end{tikzcd}$$

Proposition 13. (1) *Let X be a path-connected topological space, then the augmentation induces a (natural) isomorphism $\epsilon_*\colon H_0(X) \to \mathbb{Z}$.*

(2) *Let X be a topological space, then we have a (natural) isomorphism $H_0(X) \cong \mathbb{Z}\pi_0(X)$.*

In the second statement of this proposition we use the notation $\mathbb{Z}S$ for the free abelian group generated by the set S. Moreover, $\pi_0(X)$ denotes the set of path components of the space X. The proof of this proposition will be given in the next lecture.

Remark 14. In algebraic topology it is very convenient to make systematical use of the language of category theory. Since we did not wish to overwhelm the reader by too much of this language at the very beginning we decided to slowly develop the corresponding terminology as the course goes on. In particular, the notion of a *natural transformation between functors* will only be made precise at a later stage although they already showed up in this lecture. As a compromise we wrote '(natural) morphism' or '(natural) isomorphism' in the corresponding situations.