LECTURE 12: ISOMORPHISM BETWEEN CELLULAR AND SINGULAR HOMOLOGY

The aim of this final lecture is to show that for CW complexes we have an isomorphism between cellular and singular homology. To begin with let us recall from the previous lecture, that the n-th cellular chain group is defined by

$$C_n^{\mathrm{cell}}(X) = H_n(X^{(n)}, X^{(n-1)}).$$

and that the cellular boundary operator $C_n^{\mathrm{cell}}(X) \to C_{n-1}^{\mathrm{cell}}(X)$ is the connecting homomorphism of the triple $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$. Note that this definition of the cellular chain complex of a CW complex does not only depend on the underlying space but also on the chosen CW structure. In fact, by definition the cellular chain groups are relative homology groups of subsequent filtration steps in the skeleton filtration. Thus, one might wonder whether the resulting cellular homology is an invariant of the underlying space only (in that it would be independent of the actual choice of a CW structure). Theorem 2 tells us, in particular, that this is indeed the case.

We split off a preliminary lemma.

Lemma 1. *Let X be a CW complex. The canonical map $H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_{n+1}(X, X^{(n)})$ is surjective for every $n \geq 0$.*

Proof. For this it suffices to consider the long exact homology sequences associated to the triple $(X, X^{(n+1)}, X^{(n)})$. The relevant part of it is given by

$$H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_{n+1}(X, X^{(n)}) \to H_{n+1}(X, X^{(n+1)}).$$

But by Proposition 4 of previous lecture, the group $H_{n+1}(X, X^{(n+1)})$ is trivial, concluding the proof. □

Theorem 2. ***(Singular and cellular homology are isomorphic.)***
Let X be a CW complex. Then there is an isomorphism $H_n(X) \cong H_n^{\mathrm{cell}}(X), n \geq 0$, which is natural with respect to cellular maps.

Proof. Let us begin by identifying the *cellular cycles*, i.e., the kernel of the cellular boundary operator,

$$Z_n^{\mathrm{cell}}(X) = \ker(H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)})).$$

By definition, this boundary operator factors as

$$H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

But the second map in this factorization is injective as one easily checks using the long exact homology sequence of the pair $(X^{(n-1)}, X^{(n-2)})$ together with the fact that $H_{n-1}(X^{(n-2)})$ vanishes. This implies that $Z_n^{\mathrm{cell}}(X)$ is simply the kernel of $H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)})$. If we consider the long exact homology sequence of $(X^{(n)}, X^{(n-1)})$, then the interesting part reads as

$$H_n(X^{(n-1)}) \to H_n(X^{(n)}) \to H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}).$$

Using that $H_n(X^{(n-1)})$ is trivial, we conclude that there is a canonical isomorphism

$$H_n(X^{(n)}) \xrightarrow{\cong} Z_n^{\mathrm{cell}}(X),$$

and that this isomorphism is induced by the map $H_n(X^{(n)}) \to H_n(X^{(n)}, X^{(n-1)})$.

Let us now describe the *cellular boundaries*, i.e., the image of the cellular boundary operator,

$$B_n^{\mathrm{cell}}(X) = \mathrm{im}(H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_n(X^{(n)}, X^{(n-1)})).$$

Again, by definition this map is $H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_n(X^{(n)}) \to H_n(X^{(n)}, X^{(n-1)})$. By the first part of this proof, we know that $H_n^{\mathrm{cell}}(X)$ is canonically isomorphic to the cokernel of the first map $H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_n(X^{(n)})$, the connecting homomorphism of the pair $(X^{(n+1)}, X^{(n)})$. Recall that these connecting homomorphisms are natural with respect to maps of pairs, hence applied to the map $(X^{(n+1)}, X^{(n)}) \to (X, X^{(n)})$ this yields the following commutative diagram

$$\begin{array}{ccccccc}
H_{n+1}(X^{(n+1)}, X^{(n)}) & \longrightarrow & H_n(X^{(n)}) & & & & \\
\downarrow & & \downarrow{=} & & & & \\
H_{n+1}(X, X^{(n)}) & \longrightarrow & H_n(X^{(n)}) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, X^{(n)}),
\end{array}$$

in which the lower row is part of the long exact sequence of the pair $(X, X^{(n)})$. By Lemma 1, the vertical map on the left is surjective, and $H_n^{\mathrm{cell}}(X)$ is thus canonically isomorphic to the cokernel of $H_{n+1}(X, X^{(n)}) \to H_n(X^{(n)})$. But since $H_n(X, X^{(n)})$ vanishes, the above exact sequence allows us to conclude that $H_n^{\mathrm{cell}}(X)$ is isomorphic to $H_n(X)$. It follows from this proof that the isomorphism is compatible with cellular maps. □

This theorem allows us to deduce the following important qualitative results about homology groups which are not at all obvious from the *definition* of singular homology. Just to emphasize, let us recall that the n-th singular chain group of a space X is the free abelian group generated by all continuous maps $\Delta^n \to X$. So, a priori, it is not obvious that singular homology groups are finitely generated – even for reasonable spaces.

Corollary 3. (1) *Let X be a CW complex with finitely many cells in each dimension only. Then the homology groups $H_n(X)$ are finitely generated for all $n \geq 0$.*

(2) *Let X be an n-dimensional, finite CW complex. Then the homology groups $H_k(X)$ are trivial for $k > n$ and are finitely generated in the remaining dimensions.*

Thus, we have examples of spaces which belong to the following class.

Definition 4. A space X is **of finite type** if the singular homology group $H_k(X)$ is finitely generated for $k \geq 0$.

Recall the structure theorems for finitely generated abelian groups. Given such an abelian group, it can be written as a direct sum of a free part and its torsion part. Moreover, the free part is a direct sum of finitely many, say r, copies of the integers. This number r can be shown to be well-defined, and is called the **rank** of the finitely generated abelian group. Derived from this we obtain the following important numerical invariants of spaces of finite type.

Definition 5. Let X be a space of finite type. The k-th **Betti number** $\beta_k(X)$ of X is the rank of $H_k(X)$, i.e., we set

$$\beta_k(X) = \mathrm{rk}(H_k(X)), \quad k \geq 0.$$

If in addition only finitely many Betti numbers are non-zero, then we define its **Euler-Poincaré characteristic** $\chi(X)$ as the finite sum

$$\chi(X) = \Sigma_{k\geq 0}(1)^k \beta_k(X).$$

Until now the cellular boundary homomorphisms were only given a rather abstract description. Nevertheless, even without a more geometric understanding of these homomorphisms, we can already do some calculations in cellular homology.

Example 6. (1) The n-sphere S^n has a CW structure consisting of two cells only: one 0-dimensional cell and one n-dimensional cell. There is a unique attaching map for the n-cell which collapses the boundary S^{n-1} to a single point. Thus, for every $n \geq 2$ we have isomorphisms $C_0^{\mathrm{cell}}(S^n) \cong C_n^{\mathrm{cell}}(S^n) \cong \mathbb{Z}$. Since there are no non-trivial cellular boundary maps, we recover our earlier calculation.

(2) Recall that the complex projective space $\mathbb{C}P^n$ can be endowed with a CW structure consisting of a unique $2l$-cell for each $0 \leq l \leq n$ and no further cells. Thus the cellular chain complex is concentrated in even degrees so that all cellular boundary homomorphisms have to be zero for trivial reasons. We thus obtain the following result:

$$H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & , \quad k = 2l,\ 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Any of the generators in the top-dimension is an **orientation class** of $\mathbb{C}P^n$.

If we want to be able to calculate the cellular homology of CW complexes which have cells in subsequent dimensions, then it is helpful to have a more geometric description of the cellular boundary homomorphism. Such a description can be obtained by means of the homological degrees of self-maps of the spheres, as discussed in Lecture 8. Let us recall that the cellular chain groups are free abelian groups, i.e., we have isomorphisms $\bigoplus_{J_n} \mathbb{Z} \cong C_n^{\mathrm{cell}}(X)$ where J_n denotes the index set for the n-cells of X. Under these isomorphisms, the cellular boundary maps hence correspond to homomorphisms

$$\bigoplus_{J_n} \mathbb{Z} \cong C_n^{\mathrm{cell}}(X) \stackrel{\partial_n^{\mathrm{cell}}}{\to} C_{n-1}^{\mathrm{cell}}(X) \cong \bigoplus_{J_{n-1}} \mathbb{Z}.$$

This composition sends every n-cell σ of X to a sum

$$\sigma \mapsto \Sigma_{\tau \in J_{n-1}} z_{\sigma,\tau} \tau$$

for suitable integer coefficients $z_{\sigma,\tau}$. To conclude the description of this assignment we thus have to specify these coefficients. For that purpose, let us fix an n-cell σ and an $(n-1)$-cell τ. The n-cell σ comes with an attaching map

$$\begin{array}{ccc} S^{n-1} & \stackrel{\chi_\sigma}{\longrightarrow} & X^{(n-1)} \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X^{(n)}. \end{array}$$

Now, associated to this attaching map we can consider the following composition

$$f_{\sigma,\tau} \colon S^{n-1} \stackrel{\chi_\sigma}{\to} X^{(n-1)} \to X^{(n-1)}/X^{(n-2)} \cong \bigvee_{J_{n-1}} S^{n-1} \to S^{n-1}$$

in which the last arrow maps all copies of the spheres constantly to the base point except the one belonging to the index $\tau \in J_{n-1}$ on which the map is the identity. Thus, for each such pair of cells we obtain a pointed self-map $f_{\sigma,\tau}$ of S^{n-1} and its degree turns out to coincide with $z_{\sigma,\tau}$. Note that since S^{n-1} is compact it follows that for any n-cell σ there are only finitely many $(n-1)$-cells τ such that $f_{\sigma,\tau}$ is not the constant map. Thus, the sums in the following proposition is well-defined.

Proposition 7. *Under the above isomorphisms the cellular boundary homomorphism is given by the map*

$$\bigoplus_{J_n} \mathbb{Z} \to \bigoplus_{J_{n-1}} \mathbb{Z}\colon \quad \sigma \mapsto \Sigma_{\tau\in J_{n-1}} \deg(f_{\sigma,\tau})\tau.$$

Hence in the context of a specific CW complex in which we happen to be able to calculate all the degrees showing up in the proposition, the problem of calculating the homology of the CW complex is reduced to a purely algebraic problem.

Let us give a brief discussion the example of the real projective spaces $\mathbb{R}P^n, n \geq 0$. We begin by recalling that $\mathbb{R}P^n$ is obtained from S^n by identifying antipodal points. Hence, there are quotient maps $p = p_n\colon S^n \to \mathbb{R}P^n$. The real projective space $\mathbb{R}P^n$ can be endowed with a CW structure such that there is a unique k-cell in each dimension $0 \leq k \leq n$. One can check that the cellular boundary homomorphism $\partial\colon C_k^{\text{cell}}(\mathbb{R}P^n) \to C_{k-1}^{\text{cell}}(\mathbb{R}P^n), 0 < k \leq n$ is zero if k is odd and multiplication by 2 if k is even. From this one can derive the following calculation.

Example 8. The homology of an even-dimensional real projective space is given by

$$H_k(\mathbb{R}P^{2m}) \cong \begin{cases} \mathbb{Z} & , \quad k = 0, \\ \mathbb{Z}/2\mathbb{Z} & , \quad \text{k odd}, 0 < k < 2m \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In particular, the top-dimensional homology group $H_{2m}(\mathbb{R}P^{2m})$ is zero. The homology of odd-dimensional real projective spaces looks differently and is given by

$$H_k(\mathbb{R}P^{2m+1}) \cong \begin{cases} \mathbb{Z} & , \quad k = 0, 2m+1 \\ \mathbb{Z}/2\mathbb{Z} & , \quad \text{k odd}, 0 < k < 2m+1 \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In this case, the top-dimensional homology group is again simply a copy of the integers. Any generator of this group is called **fundamental class** of $\mathbb{R}P^{2m+1}$.

Note that these are our first examples of spaces in which the homology groups have non-trivial torsion elements. This should not be considered as something exotic but instead it is a general phenomenon. We conclude this lecture with a short outlook. There is an axiomatic approach to homology which is due to Eilenberg and Steenrod. By definition a **homology theory** consists of functors $h_n, n \geq 0$, from the category of pairs of topological spaces to abelian groups together with natural transformations (called *connecting homomorphisms*)

$$\delta\colon h_n(X, A) \to h_{n-1}(A, \emptyset), \quad n \geq 1.$$

This data has to satisfy the *long exact sequence axiom*, the *homotopy axiom*, the *excision axiom*, and the *dimension axiom*. We let you guess the precise form of the first three axioms, but we want to be specific about the dimension axiom. It asks that $h_k(*, \emptyset)$ is trivial in positive dimensions. Thus, the only possibly non-trivial homology group of the point sits in degree zero and that group $h_0(*, \emptyset)$ is referred to as the **group of coefficients** of the homology theory. So, parts of this course can be summarized by saying that singular homology theory defines a homology theory in the sense of Eilenberg-Steenrod with integral coefficients. In the sequel to this course we study closely related algebraic invariants of spaces, namely *homology groups with coefficients* and *cohomology groups*.