SECTION 1: SINGULAR HOMOLOGY WITH COEFFICIENTS

In this section we introduce a variant of singular homology, namely *singular homology with coefficients*. Recall that in the previous course we introduced singular homology groups of spaces and we established certain fundamental theorems about these invariants. Using these fundamental theorems only we were able to show that singular homology and cellular homology agree on CW complexes – a result which was already seen to be useful in specific calculations.

This suggests that there should be a purely axiomatic approach to homology theory which is based on the above-mentioned fundamental theorems. As shown by Eilenberg and Steenrod, this is indeed possible. They introduced the so-called **Eilenberg–Steenrod axioms** for homology theories. By definition a **homology theory** consists of functors $h_k, k \geq 0$, from the category of pairs of topological spaces to abelian groups together with natural transformations (called *connecting homomorphisms*)

$$\delta\colon h_k(X, Y) \to h_{k-1}(Y, \emptyset), \quad k \geq 1.$$

This data has to satisfy the *long exact sequence axiom*, the *homotopy axiom*, the *excision axiom*, and the *dimension axiom*. We let you guess the precise form of the first three axioms, but we want to be specific about the dimension axiom. It asks that $h_k(*, \emptyset)$ is trivial in positive dimensions. Thus, the only possibly non-trivial homology group of the point sits in degree zero and that abelian group $h_0(*, \emptyset)$ is referred to as the **group of coefficients** of the homology theory.

So, parts of the previous course can be summarized by saying that singular homology theory defines a *homology theory in the sense of Eilenberg–Steenrod with integral coefficients*. In this section we introduce singular homology with coefficients in an abelian group A, and basically show that this defines a homology theory in the sense of Eilenberg–Steenrod with A as group of coefficients. As a punchline, this allows us to 'work with singular homology with coefficients as in the integral case'.

Our original motivation to consider singular homology was that we wanted understand a given space X by studying formal finite sums of singular simplices in it. Such a formal sum can always be written in the form

$$\Sigma_{i=1}^{m} n_i \sigma_i \quad \text{with} \quad \sigma_i\colon \Delta^k \to X \quad \text{and} \quad n_i \in \mathbb{Z},$$

where Δ^k denotes the geometric k-simplex. Now, we want to replace the 'coefficients' $n_i \in \mathbb{Z}$ by elements of an arbitrary abelian group A, and this is formally achieved using the tensor product of abelian groups. We begin by recalling some basics about the tensor product.

Definition 1. Let A and B be abelian groups. A **tensor product** $A \otimes B$ of A and B is an abelian group $A \otimes B$ together with a bilinear map $A \times B \to A \otimes B$ which is initial in the sense that for every abelian group C and every bilinear map $A \times B \to C$ there is a unique group homomorphism $A \otimes B \to C$ such that the following diagram commutes

$$\begin{array}{ccc} A \times B & \longrightarrow & A \otimes B \\ & \searrow^{\forall} & \downarrow{\scriptstyle \exists!} \\ & & C. \end{array}$$

As with every definition by a universal property, it follows easily from the definition that tensor products are, in a certain precise sense, unique up to unique isomorphism. In the exercises you are asked to show that tensor products always exist. If $A \times B \to A \otimes B$ is 'the' universal bilinear map, then we write $a \otimes b \in A \otimes B$ for the image of (a, b). Note that the bilinearity of this map implies the manipulation rules

(1) $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$,
(2) $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$,
(3) $(na) \otimes b = n(a \otimes b) = a \otimes (nb), \quad n \in \mathbb{Z}$.

If follows from the construction that every element in $A \otimes B$ can be written as a *finite sum* $\Sigma_{i=1}^m a_i \otimes b_i$ (but, in general, not every element is of the form $a \otimes b$!).

In the following proposition we collect some fundamental properties of the tensor product construction. Once we know that tensor products exist, all these properties follow from the universal property (see the exercises). We denote by Ab the category of abelian groups and group homomorphisms.

Proposition 2. (1) *The tensor product defines a functor* $\otimes\colon \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$.
(2) *The tensor product is symmetric, i.e., there are natural isomorphisms* $A \otimes B \cong B \otimes A$.
(3) *There are isomorphisms* $0 \otimes A \cong 0 \cong A \otimes 0$ *where* 0 *is the trivial abelian group.*
(4) *The canonical map* $\bigoplus_{i \in I}(A_i \otimes B) \to (\bigoplus_{i \in I} A_i) \otimes B$ *is a natural isomorphism as is the canonical map* $\bigoplus_{i \in I}(A \otimes B_i) \to A \otimes (\bigoplus_{i \in I} B_i)$.

Example 3. (1) There are natural isomorphisms $\mathbb{Z} \otimes A \cong A \cong A \otimes \mathbb{Z}$.
(2) There are isomorphisms $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/r\mathbb{Z}$ where $r = \gcd(p, q)$ is the greatest common divisor of p and q.

Note that this example together with the above properties allows us to explicitly calculate tensor products of arbitrary finitely generated abelian groups. A further consequence of these properties is that the partial tensor product functors are *additive functors*. Before we make this notion precise let us recall that for abelian groups A, A' the set of group homomorphisms from A to A' is itself an abelian group. The addition is defined pointwise and the zero homomorphism $0\colon A \to A'$ is the neutral element. Now, the functor $- \otimes B\colon \mathsf{Ab} \to \mathsf{Ab}$ is **additive** in the following sense.

(1) For the zero map $0\colon A \to A'$ we have $0 \otimes \mathrm{id}_B = 0\colon A \otimes B \to A' \otimes B$.
(2) If $f, g\colon A \to A'$ are group homomorphisms, then

$$(f + g) \otimes \mathrm{id}_B = f \otimes \mathrm{id}_B + g \otimes \mathrm{id}_B\colon A \otimes B \to A' \otimes B.$$

Note that the first property is a consequence of the second one, but is mentioned explicitly here in order to emphasize. In the following lemma, $\mathsf{Ch}_{\geq 0}(\mathbb{Z})$ denotes the category of chain complexes of abelian groups and chain maps.

Lemma 4. *Let* A *be an abelian group and let* C, D *be chain complexes of abelian groups.*

(1) *The tensor product* $(-) \otimes A\colon \mathsf{Ab} \to \mathsf{Ab}$ *induces a functor* $(-) \otimes A\colon \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathsf{Ch}_{\geq 0}(\mathbb{Z})$ *defined by the formula* $(C \otimes A)_n = C_n \otimes A$.
(2) *If* $f, g\colon C \to D$ *are chain homotopic, then so are* $f \otimes \mathrm{id}_A, g \otimes \mathrm{id}_A\colon C \otimes A \to D \otimes A$.

Proof. The first property is immediate since $- \otimes A$ preserve zero homomorphisms so that a degreewise definition will do the job. Recall that the defining formula for chain homotopies uses only compositions and sums of homomorphisms. Since both are preserved by $- \otimes A$ also the second claim is immediate. □

Given a topological space X, we denote its singular chain complex by $C(X) \in \mathsf{Ch}_{\geq 0}(\mathbb{Z})$. Recall that $C(X)$ consists of the singular chain groups $C_k(X)$ which are free abelian groups in each degree. We refer to this by saying that $C(X)$ is **levelwise free**. If A is an abelian group, then the **singular chain complex** $C(X;A)$ of X **with coefficients in** A is given by

$$C(X;A) = C(X) \otimes A.$$

Thus, $C(X;A)$ consists of the abelian groups $C_k(X;A) = C_k(X) \otimes A, k \geq 0$, together with differentials

$$\partial \otimes \mathrm{id}_A \colon C_k(X) \otimes A \to C_{k-1}(X) \otimes A, \quad k \geq 1.$$

Definition 5. Let X be a space and let A be an abelian group. The k-th **singular homology group** $H_k(X;A)$ of X **with coefficients in** A is given by

$$H_k(X;A) = H_k(C(X;A)), \quad k \geq 0.$$

Obviously, singular homology with coefficients $H_k(-;A)$ is a functor from the category Top of topological spaces to Ab which is defined as the composition

$$H_k(-;A) \colon \mathsf{Top} \stackrel{C(-)}{\to} \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \stackrel{(-)\otimes A}{\to} \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \stackrel{H_k}{\to} \mathsf{Ab}.$$

Example 6.

(1) Since the functor $(-) \otimes \mathbb{Z} \colon \mathsf{Ab} \to \mathsf{Ab}$ is naturally isomorphic to the identity functor, we obtain natural isomorphisms

$$C(X;\mathbb{Z}) \cong C(X) \qquad \text{and} \qquad H_*(X;\mathbb{Z}) \cong H_*(X).$$

Thus, **singular homology with integral coefficients** is simply singular homology as studied in the previous course.

(2) The singular homology $H_k(*;A)$ is trivial in positive dimensions and is isomorphic to A in the case of $k = 0$.

(3) Singular homology with coefficients is 'additive'. More precisely, let X be a space and let $X_\alpha \to X, \alpha \in \pi_0(X)$, be the inclusions of its path-components. Then the canonical map

$$\bigoplus_\alpha H_k(X_\alpha;A) \to H_k(X;A), \quad k \geq 0,$$

is an isomorphism.

It is easy to define a relative version of singular homology with coefficients. If A is an abelian group and (X,Y) a pair of spaces, then the **relative singular chain complex** $C(X,Y;A)$ of (X,Y) with coefficients in A is

$$C(X,Y;A) = C(X,Y) \otimes A.$$

One might wonder if one shouldn't instead use the quotient complex of $C(X;A)$ by $C(Y;A)$ as definition for $C(X,Y;A)$, and we will soon see that both definitions would agree.

Definition 7. Let (X,Y) be a pair of spaces and let A be an abelian group. The k-th **relative singular homology** $H_k(X,Y;A)$ of (X,Y) **with coefficients** in A is

$$H_k(X,Y;A) = H_k(C(X,Y;A)), \quad k \geq 0.$$

Also relative singular homology $H_k(-;A)$ with coefficients defines a functor, which, denoting the category of pairs of spaces by Top_2, is the composition

$$H_k(-;A) \colon \mathsf{Top}_2 \stackrel{C(-)}{\to} \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \stackrel{(-)\otimes A}{\to} \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \stackrel{H_k}{\to} \mathsf{Ab}.$$

Having the basic notions in place, let us now convince ourselves that homology with coefficients shares key formal features of homology with integral coefficients. Many of them follow almost for free from the work done in the previous course. We know already that homology with coefficients defines a functor which satisfies the dimension axiom and is 'additive' in the sense of Example 6.

As a next step, let us establish the long exact sequence of a pair. For this purpose, we recall that for a pair of spaces (X, Y) there is the defining short exact sequence of chain complexes

$$0 \to C(Y) \to C(X) \to C(X, Y) \to 0.$$

One special feature of these short exact sequences is that in each dimension this sequence splits (note that we are not saying that the sequence splits as a sequence of chain complexes, i.e., the chosen sections in the various degrees do not necessarily assemble to a chain map!). In fact, this is a special case of the following lemma.

Lemma 8. *Let $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ be a short exact sequence of abelian groups such that A'' is free. Then the sequence* ***splits****, i.e., there is a homomorphism $s\colon A'' \to A$ such that $ps = \mathrm{id}_{A''}$.*

Proof. Let $\{a''_j\}_{j\in J}$ be a basis for the free abelian group A''. Since $p\colon A \to A''$ is surjective, we can choose elements $a_j \in A, j \in J$, such that $p(a_j) = a''_j$. By the universal property of free abelian groups this extends uniquely to a group homomorphism $s\colon A'' \to A$ and it is immediate that this defines a section of p. □

Recall that a functor $F\colon \mathsf{Ab} \to \mathsf{Ab}$ is additive if it satisfies the equation $F(f + g) = F(f) + F(g)$ for all group homomorphisms $f, g\colon A \to B$. In general, it is not true that an additive functor sends short exact sequences to short exact sequences. We will see counterexamples in the next section. However, a convenient fact about *split* short exact sequences of abelian groups and hence levelwise split short exact sequences of chain complexes is that they are preserved by *all additive* functors.

Lemma 9. *Let $F\colon \mathsf{Ab} \to \mathsf{Ab}$ be an additive functor.*

1. *By a levelwise application we obtain an induced functor $F\colon \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathsf{Ch}_{\geq 0}(\mathbb{Z})$.*
2. *The induced functor $F\colon \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathsf{Ch}_{\geq 0}(\mathbb{Z})$ sends chain homotopies to chain homotopies.*
3. *If $0 \to A' \to A \to A'' \to 0$ is a split short exact sequence of abelian groups then so is its image $0 \to F(A') \to F(A) \to F(A'') \to 0$. Similarly, if $0 \to C' \to C \to C'' \to 0$ is a levelwise split short exact sequence of chain complexes then the same is true for the image $0 \to F(C') \to F(C) \to F(C'') \to 0$.*

Proof. See Exercise Sheet 1. □

If we apply this lemma to $(-) \otimes A\colon \mathsf{Ab} \to \mathsf{Ab}$, then we obtain the following result.

Theorem 10. ***(Long exact sequence of a pair)***
Let (X, Y) be a pair of spaces and let A be an abelian group. Then there is a natural long exact sequence in relative homology,

$$\ldots \to H_1(X, Y; A) \to H_0(Y; A) \to H_0(X; A) \to H_0(X, Y; A) \to 0.$$

Proof. By Lemma 8, the short exact sequence $0 \to C(Y) \to C(X) \to C(X, Y) \to 0$ is levelwise split. The same is true for $0 \to C(Y; A) \to C(X; A) \to C(X, Y; A) \to 0$ by Lemma 9 applied to the additive functor $(-) \otimes A$. Thus this is, in particular, a short exact sequence of chain complexes. To conclude we invoke the existence of the natural long exact homology sequence associated to a short exact sequence of chain complexes. □

The fact that singular homology with coefficients is homotopy invariant follows immediately from the results established so far.

Theorem 11. ***(Homotopy invariance)***
Singular homology with coefficients is a homotopy invariant functor.

Proof. If $f, g\colon X \to X'$ are homotopic maps, then the induced maps $f_*, g_*\colon C(X) \to C(X')$ are chain homotopic. By Lemma 3 also $f_*, g_*\colon C(X;A) \to C(X';A)$ are chain homotopic, and these maps hence induce the same maps in homology, $f_* = g_*\colon H_k(X;A) \to H_k(X';A), k \geq 0$. $\square$

The excision property is more subtle. In fact, the way we present it here, it relies on a non-trivial, important algebraic fact. To begin with let us recall the statement of the excision theorem for integral homology. If $U \subseteq Y \subseteq X$ are spaces such that the closure of U lies in the interior of Y, then we have isomorphisms

$$H_k(X-U, Y-U) \xrightarrow{\cong} H_k(X,Y), \qquad k \geq 0.$$

More precisely, these isomorphisms are induced by the chain map $C(X-U, Y-U) \to C(X,Y)$ which in turn comes from the inclusion $(X-U, Y-U) \to (X,Y)$. Chain maps having the property that they induce isomorphisms in homology are very important in many different situations and hence deserve a special name.

Definition 12. Let $f\colon C \to D$ be a chain map.

(1) The map f is a **quasi-isomorphism** if the induced maps $f_*\colon H_k(C) \to H_k(D)$ are isomorphisms for all $k \geq 0$.
(2) The map f is a **chain homotopy equivalence** if there is a chain map $g\colon D \to C$ such that $g \circ f$ and $f \circ g$ are chain homotopic to the respective identities.

Proposition 13. *Every chain homotopy equivalence is a quasi-isomorphism.*

Proof. Let $f\colon C \to D$ be a chain homotopy equivalence and let $g\colon D \to C$ be such that $gf \simeq \mathrm{id}$ and $fg \simeq \mathrm{id}$. Then the induced maps in homology $f_*\colon H_k(C) \to H_k(D)$, $g_*\colon H_k(D) \to H_k(C)$ are inverse isomorphisms because $f_*g_* = (fg)_* = \mathrm{id}_* = \mathrm{id}$ and similarly in the other direction. The second equality holds because chain homotopic maps induce the same maps in homology. $\square$

Any chain map g as in the proof is called an **inverse chain homotopy equivalence**. One might wonder if there is also a converse to the statement of the proposition. The following example shows that in general this is not the case.

Example 14. Let us consider the following short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, where the undecorated map is the quotient map. As with every short exact sequence, this can be reinterpreted as a chain map,

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z}.
\end{array}$$

Moreover, the fact that we started with a short exact sequence precisely means that this chain map is a quasi-isomorphism. It is an exercise to show that this quasi-isomorphism is not a chain homotopy equivalence (note that $\mathbb{Z}/2\mathbb{Z}$ is torsion).

Thus, in general, not every quasi-isomorphism is a chain homotopy equivalence. One of the good properties of chain homotopy equivalences is as follows.

Lemma 15. *Let $F\colon \mathsf{Ab} \to \mathsf{Ab}$ be an additive and let $f\colon C \to D$ be a chain map. If f is a chain homotopy equivalence, then so is $F(f)\colon F(C) \to F(D)$.*

Proof. This is immediate since we already know that additive functors send chain homotopies to chain homotopies. □

This property is not enjoyed by arbitrary quasi-isomophisms as we will see in the following section. Thus, in general, additive functors do not perserve quasi-isomorphisms between arbitrary chain complexes. However, for nice chain complexes this is the case thanks to the following important theorem.

Theorem 16. *Let $C, D \in \mathsf{Ch}_{\geq 0}(\mathbb{Z})$ be levelwise free. Every quasi-isomorphism $f\colon C \to D$ is a chain homotopy equivalence.*

The proof of this theorem is non-trivial. On the next few exercise sheets there will be various steps with hints culminating in a proof of this theorem. These steps will be of independent interest, as they introduce some basic constructions from homological algebra which are used a lot.

For now let us use this theorem. With this theorem at hand, we can establish the excision property for homology with coefficients.

Theorem 17. ***(Excision property)***
Let $U \subseteq Y \subseteq X$ be spaces such that the closure of U lies in the interior of Y and let A be an abelian group. Then the map $C(X-U, Y-U; A) \to C(X, Y; A)$ is a quasi-isomorphism.

Proof. Note that the point-set topology assumptions are precisely the ones from the excision theorem for integral singular homology. Thus, the chain map $C(X-U, Y-U) \to C(X, Y)$ is a quasi-isomorphism. Since both chain complexes are levelwise free it follows from the previous theorem that $C(X-U, Y-U) \to C(X, Y)$ is actually a chain homotopy equivalence. The additive functor $- \otimes A\colon \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathsf{Ch}_{\geq 0}(\mathbb{Z})$ sends chain homotopy equivalences to chain homotopy equivalences, hence $C(X-U, Y-U; A) \to C(X, Y; A)$ is also a chain homotopy equivalence. It follows from Proposition 13 that this chain map is thus also a quasi-isomorphism. □

With this final step, we have extended the main formal properties of singular homology to singular homology with coefficients. We are now in position to discuss *cellular homology with coefficients*. Using literally the same arguments as in the integral case, one studies the behavior of singular homology with coefficients with respect to attaching cells and shows that it 'vanishes above the dimension'. Recall that these were key steps in the classical case. Instead of redoing this, we postpone this discussion until the following section as some of the facts are trivial consequences of the main theorem of that section.

Let us content ourselves by summarizing the work done so far.

Theorem 18. *Singular homology with coefficients in an abelian group A defines a homology theory in the sense of Eilenberg–Steenrod. The group of coefficients of $H_*(-;A)$ is A.*

The integral homology groups of a space are abelian groups as are the homology groups with coefficients. However, as we show next, if the coefficient group is an R-module for some commutative ring R (with unit element 1), then this structure is inherited by the homology groups. Let us recall that a **(left) R-module** is a pair $(M, \lambda\colon R \times M \to M)$ consisting of an abelian group M and a bilinear map

$$\lambda\colon R \times M \to M, (r, m) \mapsto \lambda(r, m) = rm = r \cdot m,$$

the *(left) multiplication by scalars*, such that $(r_1 r_2)m = r_1(r_2 m)$ and $1 \cdot m = m$. A homomorphism $f\colon M \to M'$ of R-modules is a group homomorphism $f\colon M \to M'$ such that $f(rm) = rf(m)$. This defines the category R-Mod of (left) R-modules. Note that a module over $R = \mathbb{Z}$ is simply an abelian group, while a module over a field is the same as a vector space.

Lemma 19. *Let A be an abelian group, R a commutative ring, and M an R-module. The map*

$$R \times A \times M \to A \times M\colon (r, a, m) \mapsto (a, rm)$$

induces an R-module structure on $A \otimes M$. This defines a functor $\otimes\colon \mathsf{Ab} \times R\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$.

Proof. For each $r \in R$, the map of sets $A \times M \overset{\mathrm{id}\times r}{\to} A \times M \to A \otimes M$ is easily seen to be bilinear. Additivity in the second variable for example amounts to observing that

$$a \otimes (r(m_1 + m_2)) = a \otimes (rm_1 + rm_2) = a \otimes rm_1 + a \otimes rm_2.$$

Thus, by the universal property of the tensor product, there is a *unique* induced group homomorphism $\lambda(r, -)\colon A \otimes M \to A \otimes M$ such that

$$\begin{array}{ccc} A \times M & \xrightarrow{\mathrm{id}\times r} & A \times M \\ \downarrow & & \downarrow \\ A \otimes M & \xrightarrow[\lambda(r,-)]{} & A \otimes M \end{array}$$

commutes. The uniqueness implies that this $\lambda\colon R \times (A \otimes M) \to A \otimes M$ defines an R-module structure and also that we obtain a functor that way. □

We leave it to the reader to check the details and also that $\otimes\colon \mathsf{Ab} \times R\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$ is a functor which is additive in both variables. In particular, given an R-module M we obtain an induced functor $- \otimes M\colon \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathsf{Ch}_{\geq 0}(R)$, where $\mathsf{Ch}_{\geq 0}(R)$ denotes the category of chain complexes of R-modules and R-linear chain maps. The homology groups of chain complex of R-modules are naturally R-modules, since homology is simply defined by passing to a subquotient ('kernel modulo image').

Corollary 20. *Let R be a commutative ring and let M be an R-module. Then singular homology with coefficients in M defines a functor*

$$H_k(-; M)\colon \mathsf{Top}_2 \to R\text{-}\mathsf{Mod}, \quad k \geq 0.$$

In particular, if $M = K$ is a field, then we obtain functors $H_k(-; K)\colon \mathsf{Top}_2 \to K\text{-}\mathsf{Vect}$ to the category K-Vect of vector spaces over K.

Proof. The first part is immediate from the above remarks. In fact, it suffices to observe that in our situation there is the following sequence of functors

$$H_k(-; M)\colon \mathsf{Top}_2 \overset{C(-)}{\to} \mathsf{Ch}_{\geq 0}(\mathbb{Z}) \overset{(-)\otimes M}{\to} \mathsf{Ch}_{\geq 0}(R) \overset{H_k}{\to} R\text{-}\mathsf{Mod}.$$

For the second statement, observe that the multiplication $K \times K \to K$ of the field turns K into a vector space over K. Thus, we can specialize the first statement to $M = K \in K$-Vect. □

The cases $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are particularly important, giving rise to **rational homology**, **real homology**, and **complex homology**, respectively. The associated homology groups in these cases are thus rational, real, or complex vector spaces. In particular, these homology groups have no torsion elements, which makes them a simpler, first approximation of the integral homology groups. Also, we want to mention that $H_*(-;\mathbb{R})$ and $H_*(-;\mathbb{C})$ are of quite some use in differential geometry and complex geometry, respectively.

The aim of the following section is to study the relation between $H_*(-;A)$ and $H_*(-;\mathbb{Z})$.