SECTION 5: FIBRATIONS AND HOMOTOPY FIBERS

In this section we will introduce two important classes of maps of spaces, namely the *Hurewicz fibrations* and the more general *Serre fibrations*, which are both obtained by imposing certain homotopy lifting properties. We will see that up to homotopy equivalence every map is a Hurewicz fibration. Moreover, associated to a Serre fibration we obtain a long exact sequence in homotopy which relates the homotopy groups of the fibre, the total space, and the base space. This sequence specializes to the long exact sequence of a pair which we already discussed in the previous lecture.

Definition 1. (1) A map $p\colon E \to X$ of spaces is said to have the **right lifting property** (RLP) with respect to a map $i\colon A \to B$ if for any two maps $f\colon A \to E$ and $g\colon B \to X$ with $pf = gi$, there exists a map $h\colon B \to E$ with $ph = g$ and $hi = f$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"'] & X \end{tikzcd}$$

(So h at the same time 'extends' f and 'lifts' g.)

(2) A map $p\colon E \to B$ of spaces is a **Serre fibration** if it has the RLP with respect to all inclusions of the form

$$I^n \times \{0\} \to I^n \times I = I^{n+1}, \qquad n \geq 0,$$

and a **Hurewicz fibration** if it has the RLP with respect to all maps of the form

$$A \times \{0\} \to A \times I$$

for any space A. (So evidently, every Hurewicz fibration is a Serre fibration.)

(3) If $p\colon E \to X$ is a map of spaces (but typically one of the two kinds of fibrations) and $x \in X$, then $p^{-1}(x) \subseteq E$ is called the **fiber of** p **over** x. If $x = x_0$ is a base point specified earlier, we just say *the* fiber of p for the fiber over x_0.

Thus, *Hurewicz* fibrations are those maps $p\colon E \to X$ which have the **homotopy lifting property** with respect to all spaces: given a homotopy $H\colon A \times I \to X$ of maps with target X and a lift $G_0\colon A \to E$ of $H_0 = H(-,0)\colon A \to X$ against the fibration $p\colon E \to X$ then this partial lift can be extended to a lift of the entire homotopy $G\colon A \times I \to E$, i.e., G satisfies $pG = H$ and $Gi = G_0$:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "G_0"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dashed, "G"'] & X \end{tikzcd}$$

By definition, the class of *Serre* fibrations is given by those maps which have the homotopy lifting property with respect to all cubes I^n.

Example 2. (1) Any projection $X \times F \to X$ is a Hurewicz fibration, as you can easily check.

(2) The evaluation map

$$\epsilon = (\epsilon_0, \epsilon_1)\colon X^I \to X \times X$$

at both end points is a Hurewicz fibration. Indeed, suppose we are given a commutative square

$$\begin{tikzcd} A \times \{0\} \arrow[r, "f"] \arrow[d, "i"'] & X^I \arrow[d, "\epsilon"] \\ A \times I \arrow[r, "g"'] \arrow[ur, dashed, "h"] & X \times X \end{tikzcd}$$

Or equivalently, we are given a map

$$\phi\colon (A \times \{0\} \times I) \cup (A \times I \times \{0,1\}) \to X$$

which we wish to extend to $A \times I \times I$. But $(\{0\} \times I) \cup (I \times \{0,1\}) = J^1 \subseteq I^2$ is a retract of I^2, and hence so is $A \times J^1 \subseteq A \times I^2$. Therefore we can simply precompose ϕ with the retraction $r\colon A \times I^2 \to A \times J^1$ to find the required extension.

(3) Let $p\colon E \to X$ and $f\colon X' \to X$ be arbitrary maps. Form the **fibered product** or **pullback**

$$E \times_X X' \quad = \quad \{(e, x') \mid p(e) = f(x')\}$$

topologized as a subspace of the product $E \times X'$. Then if $E \to X$ is a Hurewicz (or Serre) fibration, so is the induced projection $E \times_X X' \to X'$. This follows easily from the universal property of the pullback (see Exercise 1).

(4) If $E \to D \to X$ are two Hurewicz (or Serre) fibrations, then so is their composition $E \to X$ (see Exercise 2).

(5) Let (X, x_0) be a pointed space. The **path space** $P(X)$ (or more precisely, $P(X, x_0)$ if necessary) is the subspace of X^I (always with the compact-open topology) given by paths α with $\alpha(0) = x_0$. The map $\epsilon_1\colon P(X) \to X$ given by evaluation at 1 is a Hurewicz fibration. This follows by combining the previous examples (2) and (3).

(6) If $f\colon Y \to X$ is any map, the **mapping fibration** of f is the map

$$p\colon P(f) \to X$$

constructed as follows. The space $P(f)$ is the fibered product

$$P(f) \quad = \quad X^I \times_X Y \quad = \quad \{(\alpha, y) \mid \alpha(1) = f(y)\}$$

and the map p is given by $p(\alpha, y) = \alpha(0)$. We claim that p is a Hurewicz fibration. Indeed, suppose we are given a commutative diagram

$$\begin{tikzcd} A \times \{0\} \arrow[r, "u"] \arrow[d, "i"'] & P(f) \arrow[d, "p"] \\ A \times I \arrow[r, "v"'] & X \end{tikzcd}$$

Denoting by $\pi_1 \colon P(f) \to X^I$ and $\pi_2 \colon P(f) \to Y$ the two projection maps belonging to the pullback $P(f)$, we can first extend $\pi_2 \circ u$ as in:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "\pi_2 \circ u"] \arrow[d, "i"'] & Y \\ A \times I \arrow[ur, dashed, "\tilde{u}"'] & \end{tikzcd}$$

and then use example (2) to find a diagonal as in

$$\begin{tikzcd} A \times \{0\} \arrow[r, "\pi_1 \circ u"] \arrow[d] & X^I \arrow[d, "\epsilon = (\epsilon_0, \epsilon_1)"] \\ A \times I \arrow[ur, dashed, "w"'] \arrow[r, "(v, f \circ \tilde{u})"'] & X \times X \end{tikzcd}$$

Then $(w, \tilde{u}) \colon A \times I \to P(f)$ is a diagonal filling in the original diagram we are looking for. In fact, it lands in $P(f)$ because $\epsilon_1 w = f\tilde{u}$, and makes the diagram commute because $pw = \epsilon_0 w = v$ and $(w, \tilde{u})i = (\pi_1 u, \pi_2 u) = u$.

Here is a more abstract way of proving this: we showed in example (2) that $\epsilon \colon X^I \to X \times X$ is a Hurewicz fibration. Hence, by Exercise 1, so is the pullback $Q = (X \times Y) \times_{(X \times X)} X^I$ in:

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & X^I \arrow[d, "\epsilon"] \\ X \times Y \arrow[r, "\mathrm{id} \times f"'] & X \times X \end{tikzcd}$$

But then, by example (1) and Exercise 2, the composition $Q \to X \times Y \to X$ is a Hurewicz fibration as well. Now note that

$$Q \to P(f) \colon \quad (\alpha, x, y) \mapsto (\alpha, y)$$

defines a homeomorphism and that under this homeomorphism the two maps $Q \to X$ and $P(f) \to X$ are identified. Thus also the map $P(f) \to X$ is a Hurewicz fibration.

Exercise 3. Prove that the map ϕ in

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[dr, "f"'] & P(f) \arrow[d, "p"] \\ & X \end{tikzcd}$$

given by $\phi(y) = (\kappa_{f(y)}, y)$ is a homotopy equivalence which makes the diagram commute. Here, as usual, we denote by $\kappa_{f(y)}$ the constant path at $f(y)$.

This exercise thus shows that every map is 'homotopy equivalent' to a Hurewicz fibration. Motivated by this, one calls the fiber of p over x,

$$p^{-1}(x) \quad = \quad \{(\alpha, y) \mid \alpha(1) = f(y), \quad \alpha(0) = x\}$$

the **homotopy fiber** of f over x. Note that there is a canonical map from the fiber of f over x to the homotopy fiber of f over x given by a restriction of ϕ, namely:

$$f^{-1}(x) \to p^{-1}(x): \quad y \mapsto (\kappa_x, y)$$

Thus, the fiber 'sits in' the homotopy fiber while the homotopy fiber can be thought of as a 'relaxed' version of the fiber: the condition imposed on a point $y \in Y$ to lie in the fiber over x is that it has to be mapped to x by f, i.e., $f(y) = x$, while a point of the homotopy fiber is a pair (α, y) consisting of $y \in Y$ together with a path α in X 'witnessing' that y 'lies in the fiber up to homotopy'.

So far, all our examples are examples of Hurewicz fibrations. However, we will see in the next lecture that the weaker property of being a Serre fibration is a *local property*, and hence that all fiber bundles are examples of Serre fibrations. Moreover, this weaker notion suffices to establish the following theorem.

Theorem 4. ***(The long exact sequence of a Serre fibration)***
Let $p\colon (E, e_0) \to (X, x_0)$ be a map of pointed spaces with $i\colon (F, e_0) \to (E, e_0)$ being the fiber. Suppose that p is a Serre fibration. Then there is a long exact sequence of the form:

$$\ldots \to \pi_{n+1}(X, x_0) \xrightarrow{\delta} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(X, x_0) \xrightarrow{\delta} \ldots \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(X, x_0)$$

The 'connecting homomorphism' δ will be constructed explicitly in the proof. Before turning to the proof, let us deduce an immediate corollary. By considering the homotopy fiber H_f instead of the actual fiber, we see that we can obtain a long exact sequence for an *arbitrary* map f of pointed spaces.

Corollary 5. *Let $f\colon (Y, y_0) \to (X, x_0)$ be a map of pointed spaces and let H_f be its homotopy fiber. Then there is a long exact sequence of the form:*

$$\ldots \to \pi_{n+1}(X, x_0) \to \pi_n(H_f, *) \to \pi_n(Y, y_0) \xrightarrow{f_*} \pi_n(X, x_0) \to \ldots \to \pi_0(Y, y_0) \xrightarrow{f_*} \pi_0(X, x_0)$$

Proof. Apply Theorem 4 to the mapping fibration (Example 2.(6)) and use Exercise 3. □

Before entering in the proof of the theorem, we recall from the previous lecture the definition of the subspace $J^n \subseteq I^{n+1}$,

$$J^n = (I^n \times \{0\}) \cup (\partial I^n \times I) \quad \subseteq \quad \partial I^{n+1} \quad \subseteq \quad I^{n+1}.$$

Note that, by 'flattening' the sides of the cube, one can construct a homeomorphism of pairs

$$(I^{n+1}, J^n) \xrightarrow{\cong} (I^{n+1}, I^n \times \{0\}).$$

Thus, any Serre fibration also has the RLP with respect to the inclusion $J^n \subseteq I^{n+1}$. We will use this repeatedly in the proof.

Proof. (**of Theorem 4**) The main part of the proof consists in the construction of the operation δ. Let $\alpha\colon (I^n, \partial I^n) \to (X, x_0)$ represent an element of $\pi_n(X, x_0) = \pi_n(X)$. Let $\bar{e}_0\colon J^{n-1} \to E$ be the constant map with value e_0. Then the square

$$\begin{tikzcd} J^{n-1} \arrow[r, "\bar{e}_0"] \arrow[d] & E \arrow[d, "p"] \\ I^n \arrow[r, "\alpha"'] \arrow[ur, dashed, "\beta"] & X \end{tikzcd}$$

commutes, so by the definition of a Serre fibration we find a diagonal β. Then $\delta[\alpha]$ is the element of $\pi_{n-1}(F)$ represented by the map

$$\beta(-,1)\colon I^{n-1} \to F, \quad t \mapsto \beta(t,1).$$

Note that this indeed represents an element of $\pi_{n-1}(F)$, because the boundary of $I^{n-1} \times \{1\}$ is contained in J^{n-1}, and β maps the top face $I^{n-1} \times \{1\}$ into F since $p \circ \beta = \alpha$ maps it to x_0.

The first thing to check is that δ is well defined on homotopy classes. Suppose $[\alpha_0] = [\alpha_1]$, as witnessed by a homotopy $h\colon I^n \times I \to X$ from α_0 to α_1. Suppose also that we have chosen liftings β_0 and β_1 of α_0 and α_1 as above. Then we can define a map k making the solid square

$$\begin{tikzcd} \tilde{J}^n \arrow[r, "k"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "l"'] & X \end{tikzcd}$$

commute. Here $\tilde{J}n$ is the union of all the faces of I^{n+1} except $\{t_n = 1\}$. (It is like J^n except that we have interchanged the roles of t_n and t_{n+1}.) On $I^n \times \{0\}$ and $I^n \times \{1\}$ the map k is defined to be β_0 and β_1 respectively. On the faces $\{t_i = 0\}$, $\{t_i = 1\}$ $(i < n)$ and $\{t_n = 0\}$ the map k has constant value e_0. Now a diagonal l restricted to $I^{n-1} \times \{1\} \times I$ gives a homotopy from $\beta_0(-,1)$ to $\beta_1(-,1)$, and lies entirely in the fiver over x_0 because h is a homotopy relative to ∂I^n. This proves that $\beta_0(-,1)$ and $\beta_1(-,1)$ define the same element of $\pi_{n-1}(F)$. It also proves that $\delta[\alpha]$ thus defined does not depend on the choice of the filling β (Why?).

With these details about δ being well-defined out of the way, it is quite easy to prove that the sequence of the theorem is an exact sequence of pointed sets. (We write 'pointed sets' here because we haven't proved yet that δ is a homomorphism of groups for $n > 1$. We leave this to you as Exercise 7.)

Exactness at $\pi_n(E)$. Clearly $p_* \circ i_* = 0$ because $p \circ i$ is constant, so $\mathrm{im}(i_*) \subseteq \ker(p_*)$. For the reverse inclusion, suppose $\alpha\colon I^n \to E$ represents an element of $\pi_n(E)$ with $p_*[\alpha] = [p \circ \alpha] = 0$. Let $h\colon I^n \times I \to X$ be a homotopy rel ∂I^n from $p\alpha$ to the constant map on x_0. Choose a lift l in

$$\begin{tikzcd} J^n \arrow[r, "k"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "l"'] & X \end{tikzcd}$$

where $k|_{I^n \times \{0\}} = \alpha$ and k is constant e_0 on the other faces. Then $\gamma = l|_{I^n \times \{1\}}$ maps entirely into F, so represents an element $[\gamma] \in \pi_n(F)$ with $i_*[\gamma] = [i\gamma] = [\alpha]$ (by the homotopy l).

Exactness at $\pi_n(X)$. If $\beta\colon I^n \to E$ represents an element of $\pi_n(E)$ then for $\alpha = p \circ \beta$ we can take the same β as the diagonal filling in the construction of $\delta[\alpha]$. So $\delta p_*[\beta] = \beta|_{I^{n-1} \times \{1\}}$ which is constant e_0. Thus $\delta \circ p_* = 0$, or $\mathrm{im}(p_*) \subseteq \ker(\delta)$. For the converse inclusion, suppose $\alpha\colon I^n \to X$ represents an element of $\pi_n(X)$ with $\delta[\alpha] = 0$. Then for a lift β as in

$$\begin{tikzcd} J^{n-1} \arrow[r, "\bar{e}_0"] \arrow[d] & E \arrow[d, "p"] \\ I^n \arrow[r, "\alpha"'] \arrow[ur, dashed, "\beta"'] & X \end{tikzcd}$$

we have that $\beta(-,1)$ is homotopic to the constant map by a homotopy h relative to ∂I^{n-1} which maps into the fiber F. But then, stacking this homotopy h on top of β (i.e., by forming $h \circ_n \beta$), we obtain a map representing an element β' of $\pi_n(E)$. The image $p \circ \beta'$ is obviously homotopic to $p \circ \beta = \alpha$ because $p \circ h$ is constant, showing that $[\alpha]$ lies in the image of p_*.

Exactness at $\pi_{n-1}(F)$. For $\alpha\colon I^n \to X$ representing an element of $\pi_n(X)$, the map β in the construction of $\delta[\alpha] = [\beta(-,1)]$ shows that $\beta(-,1) \simeq \beta(-,0) = \bar{e}_0$ in E, so $i_*\delta[\alpha] = 0$. For the other inclusion, suppose $\gamma\colon I^{n-1} \to F$ represents an element of $\pi_{n-1}(F)$ with $i_*[\gamma] = 0$, as represented by a homotopy $h\colon I^{n-1} \times I \to E$ with $h(-,1) = \gamma$ and $h(-,0) = \bar{e}_0$. Then $\alpha = p \circ h$ represents an element of $\pi_n(X)$, and in the construction of $\delta[\alpha]$ we can choose the diagonal filling β to be identical to h, in which case $\delta[\alpha]$ is represented by γ. This shows that $\ker(i_*) \subseteq \mathrm{im}(\delta)$, and completes the proof of the theorem. □

Exercise 6. Show that the long exact sequence of a pointed pair (X, A), constructed in the previous lecture, can be obtained from this long exact sequence, by considering the mapping fibration of the inclusion $A \to X$ (see also the last exercise sheet).

Exercise 7. Prove that the connecting homomorphism $\delta\colon \pi_n(X, x_0) \to \pi_{n-1}(F, e_0)$ is a homomorphism of groups for $n \geq 2$.

Exercise 8. Let $p\colon E \to X$ be a Hurewicz fibration, and let $\alpha\colon I \to X$ be a path from x to y. Use the lifting property of $E \to X$ with respect to $p^{-1}(x) \times \{0\} \to p^{-1}(x) \times I$ to show that α induces a map $\alpha_*\colon p^{-1}(x) \to p^{-1}(y)$. Show that the homotopy class of α_* only depends on the homotopy class of α, and that this construction in fact defines a functor on the fundamental groupoid,

$$\pi(X) \to \mathsf{Ho}(\mathsf{Top}).$$

This last exercise shows in particular that the homotopy type of the fiber of a Hurewicz fibration is constant on path components. More precisely, if $p\colon E \to X$ is a Hurewicz fibration, then any path $\alpha\colon I \to X$ induces a homotopy equivalence between the fiber over $\alpha(0)$ and the fiber over $\alpha(1)$.