Fibrations

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1 Pullbacks

The **pullback** of two morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ consists of an object $X \times_A Y$ and two morphisms $X \times_A Y \longrightarrow X$ and $X \times_A Y \longrightarrow Y$, satisfying the following universal property

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_A Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & A
\end{tikzcd}$$

For small categories we can explicitly write

$$X \times_A Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

2 Fibrations

A map $p : E \longrightarrow B$ is a (**Hurewicz**) **fibration** if it satisfies the *homotopy lifting property (HLP)* with respect to all spaces. This means that if Y is a space and $h \circ i_0 = p \circ f$ in the diagram

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "h"'] \arrow[ur, dotted, "\tilde{h}"] & B
\end{tikzcd}$$

then there exists $\tilde{h}$ that makes the diagram commute.

Here $I = [0, 1]$ and $i_0 : y \longmapsto (y, 0)$.

E is the **total space**, B is the **base space** and $F_b = p^{-1}(b)$ for $b \in B$ is the **fiber over** b of the fibration p.

All fibers in a path component of B are homotopy equivalent. Thus, when B is path connected, all fibers are homotopy equivalent and we regard them as *the* **fiber** (unique up to homotopy) of the fibration p, denoted simply as F.

3 Replacing a map by a fibration

An arbitrary map $f : X \longrightarrow Y$ can be decomposed into a homotopy equivalence followed by a fibration:

$$\begin{array}{ccccc} X & \xrightarrow{\nu} & N_f & \xrightarrow{\rho} & Y \\ x & \longmapsto & (x, C_{f(x)}) & & \\ & & (x, \beta) & \longmapsto & \beta(0), \end{array}$$

ν is an homotopy equivalence, ρ a fibration, and $f = \rho \circ \nu$. Where

$$N_f = X \times_f Y^I = \{(x, \beta) \in X \times Y^I : f(x) = \beta(1)\}, \quad \text{the pullback of} \quad Y^I \xrightarrow{p_1} Y \xleftarrow{f} X,$$

is the **mapping path space** of f, with

$$Y^I = Hom(I, Y) \quad \text{and} \quad p_1 : \beta \longmapsto \beta(1),$$

and

$$C_{f(x)} : t \longmapsto f(x) \quad \text{(the constant map)}.$$

4 Homotopy Fiber

The **homotopy fiber** F_f of a map $f : X \longrightarrow Y$, is the fiber of the associated fibration $\rho : N_f \longrightarrow Y$.
Fix a basepoint $* \in Y$, take 0 to be the basepoint of I and let PY be the subset of Y^I consisting in based maps. Then

$$
\begin{aligned}
F_f &= \rho^{-1}(*) \\
&= \{(x,\beta) \in N_f : \beta \in PY\} \\
&= \{(x,\beta) \in X \times Y^I : f(x) = \beta(1) \wedge \beta(0) = *\} \\
&= X \times_f PY, \quad \text{the pullback of} \quad PY \xrightarrow[p_1]{} Y \xleftarrow[f]{} X.
\end{aligned}
$$

In this case, the universal property of the pullback is equivalent to a bijective correspondence between lifts and null-homotopies, i.e.

Proposition 4.1 Given spaces X, Y, Z and maps $f : X \longrightarrow Y$, $g : Z \longrightarrow X$, there is a (lifted) map $\tilde{g} : Z \longrightarrow F_f$ such that $\pi_1 \circ \tilde{g} = g$, where $\pi_1 : F_f \longrightarrow X$ is the canonical projection, iff $f \circ g \simeq C_*$ (null-homotopic).

$$
\begin{tikzcd}
Z \arrow[dr, dotted, "\tilde{g}"'] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & \\
& F_f \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\
& PY \arrow[r, "p_1"'] & Y
\end{tikzcd}
$$

Proof

($\Rightarrow$) Let $h = \pi_2 \circ \tilde{g} : Z \longrightarrow PY$. If we consider h as a map of $Z \times I$ into Y, then it must satisfy $h(z,0) = C_*(z) = *$ and $h(z,1) = (f \circ g)(z)$, because of the commutativity of the diagram and the definitions of the spaces involved. Thus, $h : f \circ g \simeq C_*$.

($\Leftarrow$) Let $h : f \circ g \simeq C_*$. Then $\tilde{g} = g \times h$ is the desired lift.

$\square$

5 Fibration sequences

The **fibration sequence** of a fibration $p : E \longrightarrow B$ with fiber F, is the sequence

$$F \hookrightarrow E \xrightarrow{p} B.$$

Examples

- For a map $f : X \longrightarrow Y$, we associate the fibration sequence $F_f \hookrightarrow N_f \xrightarrow{\rho} Y$.

- For a group G with a subgroup $H \preceq G$, the sequence $G/H \hookrightarrow BH \xrightarrow{\rho} BG$ is a fibration sequence, where BH and BG are the classifying spaces of H and G respectively.

6 References

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