

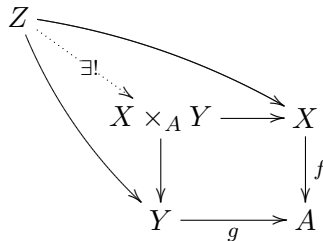
# Fibrations

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## 1 Pullbacks

The **pullback** of two morphisms  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  consists of an object  $X \times_A Y$  and two morphisms  $X \times_A Y \rightarrow X$  and  $X \times_A Y \rightarrow Y$ , satisfying the following universal property

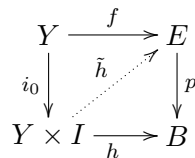


For small categories we can explicitly write

$$X \times_A Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

## 2 Fibrations

A map  $p : E \rightarrow B$  is a (**Hurewicz**) **fibration** if it satisfies the *homotopy lifting property (HLP)* with respect to all spaces. This means that if  $Y$  is a space and  $h \circ i_0 = p \circ f$  in the diagram



then there exists  $\tilde{h}$  that makes the diagram commute.

Here  $I = [0, 1]$  and  $i_0 : y \mapsto (y, 0)$ .

$E$  is the **total space**,  $B$  is the **base space** and  $F_b = p^{-1}(b)$  for  $b \in B$  is the **fiber over  $b$**  of the fibration  $p$ .

All fibers in a path component of  $B$  are homotopy equivalent. Thus, when  $B$  is path connected, all fibers are homotopy equivalent and we regard them as *the fiber* (unique up to homotopy) of the fibration  $p$ , denoted simply as  $F$ .

### 3 Replacing a map by a fibration

An arbitrary map  $f : X \rightarrow Y$  can be decomposed into a homotopy equivalence followed by a fibration:

$$\begin{array}{ccccc} X & \xrightarrow{\nu} & N_f & \xrightarrow{\rho} & Y \\ x & \mapsto & (x, C_{f(x)}) & & \\ & & (x, \beta) & \mapsto & \beta(0), \end{array}$$

$\nu$  is an homotopy equivalence,  $\rho$  a fibration, and  $f = \rho \circ \nu$ . Where

$$N_f = X \times_f Y^I = \{(x, \beta) \in X \times Y^I : f(x) = \beta(1)\}, \quad \text{the pullback of } Y^I \xrightarrow{p_1} Y \xleftarrow{f} X,$$

is the **mapping path space** of  $f$ , with

$$Y^I = \text{Hom}(I, Y) \quad \text{and} \quad p_1 : \beta \mapsto \beta(1),$$

and

$$C_{f(x)} : t \mapsto f(x) \quad (\text{the constant map}).$$

### 4 Homotopy Fiber

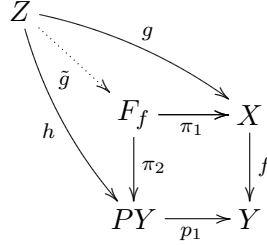
The **homotopy fiber**  $F_f$  of a map  $f : X \rightarrow Y$ , is the fiber of the associated fibration  $\rho : N_f \rightarrow Y$ .

Fix a basepoint  $* \in Y$ , take 0 to be the basepoint of  $I$  and let  $PY$  be the subset of  $Y^I$  consisting in based maps. Then

$$\begin{aligned}
F_f &= \rho^{-1}(*) \\
&= \{(x, \beta) \in N_f : \beta \in PY\} \\
&= \{(x, \beta) \in X \times Y^I : f(x) = \beta(1) \wedge \beta(0) = *\} \\
&= X \times_f PY, \quad \text{the pullback of } PY \xrightarrow{p_1} Y \xleftarrow{f} X.
\end{aligned}$$

In this case, the universal property of the pullback is equivalent to a bijective correspondence between lifts and null-homotopies, i.e.

**Proposition 4.1** Given spaces  $X, Y, Z$  and maps  $f : X \rightarrow Y$ ,  $g : Z \rightarrow X$ , there is a (lifted) map  $\tilde{g} : Z \rightarrow F_f$  such that  $\pi_1 \circ \tilde{g} = g$ , where  $\pi_1 : F_f \rightarrow X$  is the canonical projection, iff  $f \circ g \simeq C_*$  (null-homotopic).



PROOF

( $\Rightarrow$ ) Let  $h = \pi_2 \circ \tilde{g} : Z \rightarrow PY$ . If we consider  $h$  as a map of  $Z \times I$  into  $Y$ , then it must satisfy  $h(z, 0) = C_*(z) = *$  and  $h(z, 1) = (f \circ g)(z)$ , because of the commutativity of the diagram and the definitions of the spaces involved. Thus,  $h : f \circ g \simeq C_*$ .

( $\Leftarrow$ ) Let  $h : f \circ g \simeq C_*$ . Then  $\tilde{g} = g \times h$  is the desired lift.

□

## 5 Fibration sequences

The **fibration sequence** of a fibration  $p : E \rightarrow B$  with fiber  $F$ , is the sequence

$$F \hookrightarrow E \xrightarrow{p} B.$$

EXAMPLES

- For a map  $f : X \rightarrow Y$ , we associate the fibration sequence  $F_f \hookrightarrow N_f \xrightarrow{\rho} Y$ .

- For a group  $G$  with a subgroup  $H \preceq G$ , the sequence  $G/H \hookrightarrow BH \xrightarrow{\rho} BG$  is a fibration sequence, where  $BH$  and  $BG$  are the classifying spaces of  $H$  and  $G$  respectively.

## 6 References

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[Ben] D. J. Benson. *Representations and cohomology II: Cohomology of groups and modules*. Section 1.6. Cambridge studies in advanced mathematics 31, Cambridge University Press. Cambridge, 1991.

[Beh] M. Behrens. *Lecture 6: Pushouts and Pullbacks, the Homotopy Fiber*. Lecture notes of the course: Algebraic Topology II. MIT. 2006.