Lecture Notes in Algebraic Topology

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of the constructions we will give are set-theoretically simple, the issue of how to appropriately topologize these sets can become a nuisance. The category of compactly generated spaces is a framework which permits one to make such constructions without worrying about these technical issues. The reference for the material in this section is Steenrod's paper "A convenient category of topological spaces" [**38**].

Definition 6.1. A topological space X is said to be *compactly generated* if X is Hausdorff and if a subset $A \subset X$ is closed if and only if $A \cap C$ is closed for every compact $C \subset X$.

Examples of compactly generated spaces include:

1. locally compact Hausdorff spaces (e.g. manifolds),
2. metric spaces, and
3. CW-complexes with finitely many cells in each dimension.

We will use the notation $\mathcal{K}$ for the category of compactly generated spaces. (This is taken as a *full* subcategory of the category of all topological spaces, i.e. every continuous function between compactly generated spaces is a morphism in $\mathcal{K}$.)

Any Hausdorff space can be turned into a compactly generated space by the following trick.

Definition 6.2. If X is Hausdorff, let $k(X)$ be the set X with the new topology defined by declaring a subset $A \subset X$ to be closed in $k(X)$ if and only if $A \cap C$ is closed in X for all $C \subset X$ compact.

Exercise 84. Show that $k(X)$ is compactly generated.

Thus $k(X)$ is the underlying set of X topologized with more (closed and hence more) open sets than X. This construction defines a *functor*

$$k : \mathcal{T}_2 \to \mathcal{K}$$

from the category $\mathcal{T}_2$ of Hausdorff spaces to the category $\mathcal{K}$ of compactly generated spaces.

Exercise 85. Show that k is a right adjoint for the inclusion functor $i : \mathcal{K} \to \mathcal{T}_2$. You will end up having to verify several of the facts below.

6.1.1. Basic facts about compactly generated spaces.

1. If $X \in \mathcal{K}$, then $k(X) = X$.
2. If $f : X \to Y$ is a function, then $k(f) : k(X) \to k(Y)$ is continuous if and only if $f|_C : C \to Y$ is continuous for each compact $C \subset X$.
3. Let $C(X, Y)$ denote the set of continuous functions from X to Y. Then $k_* : C(X, k(Y)) \to C(X, Y)$ is a bijection if X is in $\mathcal{K}$.

4. The singular chain complexes of a Hausdorff space Y and the space $k(Y)$ are the same.
5. The homotopy groups (see Definition 6.43) of Y and $k(Y)$ are the same.
6. Suppose that $X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ is an expanding sequence of compactly generated spaces so that X_n is closed in X_{n+1}. Topologize the union $X = \cup_n X_n$ by defining a subset $C \subset X$ to be closed if $C \cap X_n$ is closed for each n. Then if X is Hausdorff, it is compactly generated. In this case every compact subset of X is contained in some X_n.

6.1.2. Products in $\mathcal{K}$. Unfortunately, the product of compactly generated spaces need not be compactly generated. However, this causes little concern, as we now see.

Definition 6.3. Let X, Y be compactly generated spaces. The *categorical product* of X and Y is the space $k(X \times Y)$.

The following useful facts hold about the categorical product.

1. $k(X \times Y)$ is in fact a product in the category $\mathcal{K}$.
2. If X is locally compact and Y is compactly generated, then $X \times Y = k(X \times Y)$. In particular, $I \times Y = k(I \times Y)$. Thus the notion of homotopy is unchanged.

From now on, if X and Y are compactly generated, we will denote $k(X \times Y)$ by $X \times Y$.

6.1.3. Function spaces. The standard way to topologize the set of functions $C(X, Y)$ is to use the compact-open topology.

Definition 6.4. If X and Y are compactly generated spaces, let $C(X, Y)$ denote the set of continuous functions from X to Y, topologized with the *compact-open topology.* This topology has as a subbasis sets of the form

$$U(K, W) = \{f \in C(X, Y) | f(K) \subset W\}$$

where K is a compact set in X and W an open set in Y.

If Y is a metric space, this is the notion, familiar from complex analysis, of uniform convergence on compact sets. Unfortunately, even for compactly generated spaces X and Y, $C(X, Y)$ need not be compactly generated. We know how to handle this problem: define

$$\mathrm{Map}(X, Y) \; = \; k(C(X, Y)).$$

As a *set*, $\mathrm{Map}(X, Y)$ is the set of continuous maps from X to Y, but its topology is slightly different from the compact open topology.

Theorem 6.5 (adjoint theorem)**.** *For X, Y, and Z compactly generated, $f(x,y) \mapsto \tilde{f}(x)(y)$ gives a homeomorphism*

$$\mathrm{Map}((X \times Y), Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

Thus $- \times Y$ and $\mathrm{Map}(Y, -)$ are adjoint functors from $\mathcal{K}$ to $\mathcal{K}$. □

The following useful properties of $\mathrm{Map}(X,Y)$ hold.

1. Let $e : \mathrm{Map}(X,Y) \times X \to Y$ be the *evaluation* $e(f,x) = f(x)$. Then if $X, Y \in \mathcal{K}$, e is continuous.
2. If $X, Y, Z \in \mathcal{K}$, then:
   (a) $\mathrm{Map}(X, Y \times Z)$ is homeomorphic to $\mathrm{Map}(X,Y) \times \mathrm{Map}(X,Z)$,
   (b) Composition defines a continuous map
   $$\mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z).$$

We will also use the notation $\mathrm{Map}(X,A;Y,B)$ to denote the subspace of $\mathrm{Map}(X,Y)$ consisting of those functions $f : X \to Y$ which satisfy $f(A) \subset B$. A variant of this notation is $\mathrm{Map}(X, x_0; Y, y_0)$ denoting the subspace of basepoint preserving functions.

6.1.4. Quotient maps. We discuss yet another convenient property of compactly generated spaces. For topological spaces, one can give an example of quotient maps $p : W \to Y$ and $q : X \to Z$ so that $p \times q : W \times X \to Y \times Z$ is not a quotient map. However, one can show the following.

Theorem 6.6.

1. *If $p : W \to Y$ and $q : X \to Z$ are quotient maps, and X and Z are locally compact Hausdorff, then $p \times q$ is a quotient map.*
2. *If $p : W \to Y$ and $q : X \to Z$ are quotient maps and all space are compactly generated, then $p \times q$ is a quotient map, provided we use the categorical product.* □

From now on, *we assume all spaces are compactly generated.* If we ever meet a space which is not compactly generated, we immediately apply k. Thus, for example, if X and Y are Hausdorff spaces, then by our convention $X \times Y$ really means $k(k(X) \times k(Y))$. By this convention, we lose no information concerning homology and homotopy, but we gain the adjoint theorem.

6.2. Fibrations

There are two kinds of maps of fundamental importance in algebraic topology; fibrations and cofibrations. Geometrically, fibrations are more complicated than cofibrations. However, your garden variety fibration tends to be a fiber bundle, and fiber bundles over paracompact spaces are always fibrations, so that we have seen many examples so far.

Definition 6.7. A continuous map $p : E \to B$ is a *fibration* if it has the *homotopy lifting property (HLP)*; i.e. the problem

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{g}"] \arrow[d] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "G"'] \arrow[ur, dotted, "\tilde{G}"] & B \end{tikzcd}$$

has a solution for every space Y.

In other words, given the continuous maps p, G, $\tilde{g}$, and the inclusion $Y \times \{0\} \to Y \times I$, the problem is to find a continuous map $\tilde{G}$ making the diagram commute.

Remark. Recall that whenever a commutative diagram is given with one dotted arrow, we consider it as a problem whose solution is a map which can be substituted for the dashed arrow to give a commutative diagram.

A covering map is a fibration. In studying covering space theory this fact is called the covering homotopy theorem. For covering maps the lifting is unique, but this is not true for an arbitrary fibration.

Exercise 86. Show that the projection to the first factor $p : B \times F \to B$ is a fibration. Show by example that the liftings need not be unique.

The following theorem of Hurewicz says that if a map is locally a fibration, then it is so globally.

Theorem 6.8. *Let $p : E \to B$ be a continuous map. Suppose that B is paracompact and suppose that there exists an open cover $\{U_\alpha\}$ of B so that $p : p^{-1}(U_\alpha) \to U_\alpha$ is a fibration for each U_α.*

Then $p : E \to B$ is a fibration. □

Proving this theorem is one of the projects for Chapter 4. The corollary of most consequence for us is the following.

Corollary 6.9. *If $p : E \to B$ is a fiber bundle over a paracompact space B, then p is a fibration.*

Proof. Exercise 86 says that the projection $U \times F \to U$ is a fibration. Since fiber bundles have this local product structure, Theorem 6.8 implies that a fiber bundle is a fibration. □

Exercise 87. Give an example of a fibration which is not a fiber bundle.

Maps between fibrations are analogous to (and simpler than) maps of fiber bundles.

Definition 6.10. If $p : E \to B$ and $p' : E' \to B'$ are fibrations, then a *map of fibrations* is a pair of maps $f : B \to B'$, $\tilde{f} : E \to E'$ so that the diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & E' \arrow[d] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

commutes.

Pullbacks make sense and exist in the world of fibrations.

Definition 6.11. If $p : E \to B$ is a fibration, and $f : X \to B$ a continuous map, define the *pullback of* $p : E \to B$ *by* f to be the map $f^*(E) \to X$ where

$$f^*(E) = \{(x,e) \in X \times E | f(x) = p(e)\} \subset X \times E$$

and the map $f^*(E) \to B$ is the restriction of the projection $X \times E \to X$.

The following exercise is a direct consequence of the universal property of pullbacks.

Exercise 88. Show that $f^*(E) \to X$ is a fibration.

The following notation will be in effect for the rest of the book. If $H : Y \times I \to B$ is a homotopy, then $H_t : Y \to B$ is the homotopy at time t, i.e.

$$H_t(y) = H(y, t).$$

6.3. The fiber of a fibration

A fibration need not be a fiber bundle. Indeed, the definition of a fibration is less rigid than that of a fiber bundle and it is not hard to alter a fiber bundle slightly to get a fibration which is not locally trivial. Nevertheless, a fibration has a well defined fiber *up to homotopy.* The following theorem asserts this, and also states that a fibration has a substitute for the structure group of a fiber bundle, namely the group of homotopy classes of self-homotopy equivalences of the fiber.

It is perhaps at first surprising that the homotopy lifting property in itself is sufficient to endow a map with the structure of a "fiber bundle up to homotopy". But as we will see, the notion of a fibration is central in studying spaces up to homotopy.

Theorem 6.12. *Let $p : E \to B$ be a fibration. Assume B is path connected.*

Then all fibers $E_b = p^{-1}(b)$ are homotopy equivalent. Moreover every path $\alpha : I \to B$ defines a homotopy class α_* of homotopy equivalences $E_{\alpha(0)} \to E_{\alpha(1)}$ which depends only on the homotopy class of α rel endpoints, in such a way that multiplication of paths *corresponds to* composition of homotopy equivalences.

In particular, there exists a well-defined group homomorphism

$$[\alpha] \mapsto (\alpha^{-1})_*$$

$\pi_1(B, b_0) \to$ *Homotopy classes of self-homotopy equivalences of E_{b_0}.*

Remark. The reason why we use $\alpha \mapsto (\alpha^{-1})_*$ instead of $\alpha \mapsto \alpha_*$ is because by convention, multiplication of paths in B is defined so that $\alpha\beta$ means first follow α, then β. This implies that $(\alpha\beta)_* = \beta_* \circ \alpha_*$, and so we use the inverse to turn this anti-homomorphism into a homomorphism.

Proof. Let $b_0, b_1 \in B$ and let α be a path in B from b_0 to b_1. The inclusion $E_{b_0} \hookrightarrow E$ completes to a diagram

$$\begin{tikzcd} E_{b_0} \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ E_{b_0} \times I \arrow[r, "H"'] & B \end{tikzcd}$$

where $H(e,t) = \alpha(t)$. Since $E \to B$ is a fibration, H lifts to E, i.e. there exists a map $\widetilde{H}$ such that

$$\begin{tikzcd} E_{b_0} \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ E_{b_0} \times I \arrow[r, "H"'] \arrow[ur, "\widetilde{H}"] & B \end{tikzcd}$$

commutes.

Notice that the homotopy at time $t = 0$, $\widetilde{H}_0 : E_{b_0} \to E$ is just the inclusion of the fiber E_{b_0} in E. Furthermore, $p \circ \widetilde{H}_t$ is the constant map at $\alpha(t)$, so the homotopy $\widetilde{H}$ at time $t = 1$ is a map $\widetilde{H}_1 : E_{b_0} \to E_{b_1}$. We will let $\alpha_* = [\widetilde{H}_1]$ denote the homotopy class of this map. Since $\widetilde{H}$ is not unique, we need to show that another choice of lift gives a homotopic map. We will in fact show something more general. Suppose $\alpha' : I \to B$ is another path homotopic to α rel end points. Then as before, we obtain a solution $\widetilde{H}'$ to the problem

$$\begin{tikzcd} E_{b_0} \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ E_{b_0} \times I \arrow[r, "H'"'] \arrow[ur, "\widetilde{H}'"'] & B \end{tikzcd}$$

(where $H' = \alpha' \circ \mathrm{proj}_I$) and hence a map $\widetilde{H}'_1 : E_{b_0} \to E_{b_1}$.

Claim. *$\widetilde{H}_1$ is homotopic to $\widetilde{H}'_1$.*

Proof of Claim. Since α is homotopic rel end points to α', there exists a map $\Lambda : E_{b_0} \times I \times I \to B$ such that

$$\Lambda(e,s,t) = F(s,t)$$

where $F(s,t)$ is a homotopy rel end points of α to α'. (So $F_0 = \alpha$ and $F_1 = \alpha'$.) The solutions $\widetilde{H}$ and $\widetilde{H}'$ constructed above give a diagram

$$\begin{tikzcd} (E_{b_0} \times I) \times \{0,1\} \cup (E_{b_0} \times \{0\}) \times I \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d, "p"] \\ (E_{b_0} \times I) \times I \arrow[r, "\Lambda"'] & B \end{tikzcd}$$

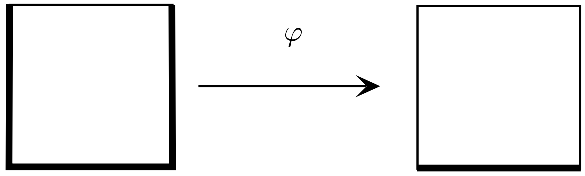
where

$$\Gamma(e,s,0) = \widetilde{H}(e,s)$$

$$\Gamma(e,s,1) = \widetilde{H}'(e,s), \text{ and,}$$

$$\Gamma(e,0,t) = e.$$

Let $U = I \times \{0,1\} \cup \{0\} \times I \subset I \times I$ There exists a homeomorphism $\varphi : I^2 \to I^2$ taking U to $I \times \{0\}$ as indicated in the following picture.



Thus the diagram

$$\begin{tikzcd}
E_{b_0} \times I \times \{0\} \arrow[d] & E_{b_0} \times U \arrow[l, "\mathrm{Id}\times\varphi"'] \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d, "p"] \\
E_{b_0} \times I \times I & E_{b_0} \times I^2 \arrow[l, "\varphi"] \arrow[r, "\Lambda"'] & B
\end{tikzcd}$$

has the left two horizontal maps homeomorphisms. Since the homotopy lifting property applies to the outside square, there exists a lift $\widetilde{\Lambda} : E_{b_0} \times I^2 \to E$ so that

$$\begin{tikzcd}
E_{b_0} \times U \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d, "p"] \\
E_{b_0} \times I^2 \arrow[ur, "\widetilde{\Lambda}"] \arrow[r, "\Lambda"'] & B
\end{tikzcd}$$

commutes.

But then $\widetilde{\Lambda}$ is a homotopy from $\widetilde{H} : E_{b_0} \times I \to E$ to $\widetilde{H}' : E_{b_0} \times I \to E$. Restricting to $E_{b_0} \times \{1\}$ we obtain a homotopy from $\widetilde{H}_1$ to $\widetilde{H}'_1$. Thus the homotopy class $\alpha_* = [\widetilde{H}_1]$ depends only on the homotopy class of α rel end points, establishing the claim.

Clearly $(\alpha\beta)_* = \beta_* \circ \alpha_*$ if $\beta(0) = \alpha(1)$. In particular, if $\beta = \alpha^{-1}$ then $(\mathrm{const})_* = \beta_* \circ \alpha_*$, where const denotes the constant path at b_0. But clearly

$$(\mathrm{const})_* = [\mathrm{Id}_{E_{b_0}}]$$

Thus β_* is a homotopy inverse of α_*.

This shows that α_* is a homotopy equivalence, and since B is path connected, all fibers are homotopy equivalent.

Applying this construction to $\alpha \in \pi_1(B,b_0)$ we see that α_* defines a homotopy equivalence of E_{b_0}, and products of loops correspond to composites of homotopy equivalences. The following exercise completes the proof. □

Exercise 89. Show that the set of homotopy classes of homotopy equivalences of a space X forms a group under composition. That is, show that multiplication and taking inverses is well defined.

Theorem 6.12 asserts that the fibers $p^{-1}(b) = E_b$ for $b \in B$ are homotopy equivalent. Thus we will abuse terminology slightly and refer to any space in the homotopy equivalence class of the space E_b for any $b \in B$ as *the fiber of the fibration* $p : E \to B$.

Since homotopy equivalences induce isomorphisms in homology or cohomology, a fibration with fiber F gives rise to local coefficients systems whose fiber is the homology or cohomology of F, as the next corollary asserts.

Corollary 6.13. *Let $p : E \to B$ be a fibration and let $F = p^{-1}(b_0)$. Then p gives rise to local coefficient systems over B with fiber $H_n(F;M)$ or $H^n(F;M)$ for any n and any coefficient group M. These local coefficients are obtained from the representations via the composite homomorphism*

$$\pi_1(B,b_0) \to \left\{ \begin{array}{l} \textit{Homotopy classes of self-homotopy} \\ \textit{equivalences } F \to F \end{array} \right\} \to \operatorname{Aut}(A)$$

where $A = H_n(F;M)$ or $A = H^n(F;M)$.

Proof. The maps $f_* : H_n(F;M) \to H_n(F;M)$ and $f^* : H^n(F;M) \to H^n(F;M)$ induced by a homotopy equivalence $f : F \to F$ are isomorphisms which depend only on the homotopy class of f. Thus there is a function from the group of homotopy classes of homotopy equivalences of F to the group of automorphisms of A. This is easily seen to be a homomorphism. The corollary follows. □

We see that a fibration gives rise to many local coefficient systems, by taking homology or cohomology of the fiber. More generally one obtains a local coefficient system given any homotopy functor from spaces to abelian groups (or R-modules), such as the generalized homology theories which we introduce in Chapter 8.

With some extra hypotheses one can also apply this to homotopy functors on the category of based spaces. For example, we will see below that if F is simply connected, or more generally "simple," then taking homotopy groups $\pi_n F$ also gives rise to a local coefficient system. For now however, observe that the homotopy equivalences constructed by Theorem 6.12 need not preserve base points.

6.4. Path space fibrations

An important family of fibrations are the path space fibrations. They will be useful in replacing arbitrary maps by fibrations and then in extending a fibration to a "fiber sequence".

Definition 6.14. Let (Y, y_0) be a based space. The *path space* $P_{y_0}Y$ is the space of paths in Y starting at y_0, i.e.

$$P_{y_0}Y = \operatorname{Map}(I,0;Y,y_0) \subset \operatorname{Map}(I,Y),$$

topologized as in the previous subsection, i.e. as a compactly generated space. The *loop space* $\Omega_{y_0}Y$ is the space of all loops in Y based at y_0, i.e.

$$\Omega_{y_0}Y = \mathrm{Map}(I,\{0,1\};Y,\{y_0\}).$$

Often the subscript y_0 is omitted in the above notation. Let $Y^I = \mathrm{Map}(I,Y)$. This is called the *free path space.* Let $p : Y^I \to Y$ be the evaluation at the end point of a path: $p(\alpha) = \alpha(1)$.

By our conventions on topologies, $p : Y^I \to Y$ is continuous. The restriction of p to $P_{y_0}Y$ is also continuous.

Exercise 90. Let y_0, y_1 be two points in a path-connected space Y. Prove that $\Omega_{y_0}Y$ and $\Omega_{y_1}Y$ are homotopy equivalent.

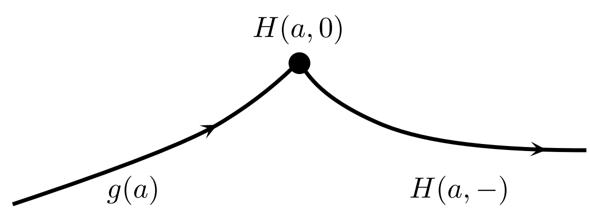
Theorem 6.15.

1. *The map $p : Y^I \to Y$, where $p(\alpha) = \alpha(1)$, is a fibration. Its fiber over y_0 is the space of paths which end at y_0, a space homeomorphic to $P_{y_0}Y$.*
2. *The map $p : P_{y_0}Y \to Y$ is a fibration. Its fiber over y_0 is the loop space $\Omega_{y_0}Y$.*
3. *The free path space Y^I is homotopy equivalent to Y. The projection $p : Y^I \to Y$ is a homotopy equivalence.*
4. *The space of paths in Y starting at y_0, $P_{y_0}Y$, is contractible.*

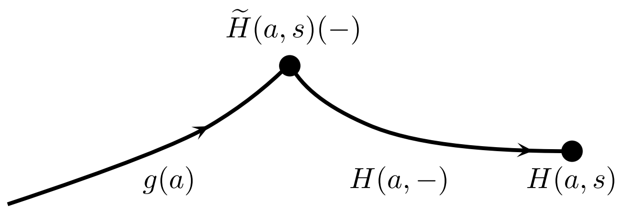
Proof. 1. Let A be a space, and suppose a homotopy lifting problem

$$\begin{tikzcd} A \times \{0\} \arrow[r, "g"] \arrow[d] & Y^I \arrow[d, "p"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & Y \end{tikzcd}$$

is given. We write $g(a)$ instead of $g(a,0)$. For each $a \in A$, $g(a)$ is a path in Y which ends at $p(g(a)) = H(a,0)$. This point is the start of the path $H(a,-)$.



We will define $\widetilde{H}(a,s)(t)$ to be a path running along the path $g(a)$ and then part way along $H(a,-)$, ending at $H(a,s)$.



Define

$$\widetilde{H}(a,s)(t) = \begin{cases} g(a)((1+s)t) & \text{if } 0 \leq t \leq 1/(1+s), \\ H(a,((1+s)t-1) & \text{if } 1/(1+s) \leq t \leq 1. \end{cases}$$

Then $\widetilde{H}(a,s)(t)$ is continuous as a function of (a,s,t), so $\widetilde{H}(a,s) \in Y^I$ and by our choice of topologies $\widetilde{H} : A \times I \to Y^I$ is continuous. Also $\widetilde{H}(a,0) = g(a)$ and $p(\widetilde{H}(a,s)) = \widetilde{H}(a,s)(1) = H(a,s)$. Thus the lifting problem is solved and so $p : P_{y_0}Y \to Y$ is a fibration. The fiber $p^{-1}(y_0)$ consists of all paths ending at y_0 and the path space $P_{y_0}Y$ consists of all paths starting at y_0. A homeomorphism is given by

$$\alpha(t) \mapsto \overline{\alpha}(t) = \alpha(1-t).$$

This proves 1.

2. has the same proof; the fact that $g(a)$ *starts at* y_0 means that $\widetilde{H}(a,s)$ also starts at y_0.

3. Let $i : Y \to Y^I$ be the map taking y to the constant path at y. Then $p \circ i = \mathrm{Id}_Y$. Let $F : Y^I \times I \to Y^I$ be given by

$$F(\alpha,s)(t) = \alpha(s+t-st).$$

Then $F(\alpha,0) = \alpha$ and $F(\alpha,1)$ is the constant path at $\alpha(1)$ which in turn equals $i \circ p(\alpha)$. Thus F shows that the identity is homotopic to $i \circ p$. Hence p and i are homotopy inverses.

4. has the same proof as 3. □

6.5. Fiber homotopy

Recall a map of fibrations ($p : E \to B$) to ($p' : E' \to B'$) is a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Definition 6.16. A *fiber homotopy* between two morphisms $(\tilde{f}_i, f_i)$ $i = 0, 1$ of fibrations is a commutative diagram

$$\begin{tikzcd} E \times I \arrow[r, "\widetilde{H}"] \arrow[d, "p\times\mathrm{Id}"'] & E' \arrow[d, "p'"] \\ B \times I \arrow[r, "H"'] & B' \end{tikzcd}$$

with $H_0 = f_0$, $H_1 = f_1$, $\widetilde{H}_0 = \tilde{f}_0$, and $\widetilde{H}_1 = \tilde{f}_1$.

Given two fibrations over B, $p : E \to B$ and $p' : E' \to B$, we say they have the same *fiber homotopy type* if there exists a map $\tilde{f}$ from E to E' covering the identity map of B, and a map $\tilde{g}$ from E' to E covering the identity map of B, such that the composites

$$\begin{tikzcd} E \arrow[rr, "\tilde{g}\circ\tilde{f}"] \arrow[dr] & & E \arrow[dl] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[rr, "\tilde{f}\circ\tilde{g}"] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd}$$

are each fiber homotopic to the identity via a homotopy which is the identity on B (i.e. there exists $\widetilde{H} : E \times I \to E$ such that $p(\widetilde{H}(e,t)) = p(e)$, $\widetilde{H}_0 = \tilde{g} \circ \tilde{f}$, and $\widetilde{H}_1 = \mathrm{Id}_E$. Similarly for $\tilde{f} \circ \tilde{g}$). One says that $\tilde{f}$ and $\tilde{g}$ are *fiber homotopy equivalences.*

Notice that a fiber homotopy equivalence $\tilde{f} : E \to E'$ induces a homotopy equivalence $E_{b_0} \to E'_{b_0}$ on fibers.

6.6. Replacing a map by a fibration

Let $f : X \to Y$ be a continuous map. We will replace X by a homotopy equivalent space P_f and obtain a map $P_f \to Y$ which is a fibration. In short, every map is equivalent to a fibration. If f is a fibration to begin with, then the construction gives a fiber homotopy equivalent fibration. We assume that Y is path-connected and X is non-empty.

Let $q : Y^I \to Y$ be the path space fibration, with $q(\alpha) = \alpha(0)$; evaluation at the starting point.

Definition 6.17. The pullback $P_f = f^*(Y^I)$ of the path space fibration along f is called the *mapping path space.*

(6.1)
$$\begin{tikzcd} P_f = f^*(Y^I) \arrow[r] \arrow[d] & Y^I \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

An element of P_f is a pair (x, α) where α is a path in Y and x is a point in X which maps via f to the starting point of α.

The *mapping path fibration*

$$p : P_f \to Y$$

is obtaining by evaluating at the end point

$$p(x, \alpha) = \alpha(1).$$

Theorem 6.18. *Suppose that $f : X \to Y$ is a continuous map.*

1. *There exists a homotopy equivalence $h : X \to P_f$ so that the diagram*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & P_f \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

   commutes.
2. *The map $p : P_f \to Y$ is a fibration.*
3. *If $f : X \to Y$ is a fibration, then h is a fiber homotopy equivalence.*

Proof. 1. Let $h : X \to P_f$ be the map

$$h(x) = (x, \mathrm{const}_{f(x)})$$

where $\mathrm{const}_{f(x)}$ means the constant path at $f(x)$. Then $f = p \circ h$, so the triangle commutes. The homotopy inverse of h is $p_1 : P_f \to X$, projection on the X-component. Then $p_1 \circ h = \mathrm{Id}_X$. The homotopy from $h \circ p_1$ to Id_{P_f} is given by

$$F((x, \alpha), s) = (x, \alpha_s),$$

where α_s is the path $s \mapsto \alpha(st)$ (We have embedded X in P_f via h, and have given a deformation retract of P_f to X by contracting a path to its starting point.)

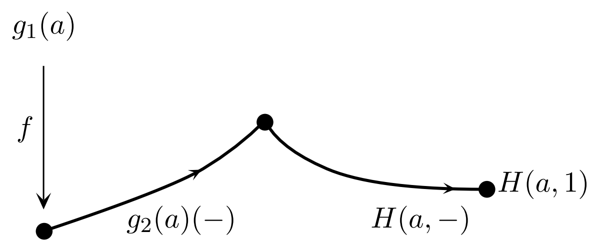
2. Let the homotopy lifting problem

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "g"] \arrow[d] & P_f \arrow[d, "p"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & Y
\end{tikzcd}$$

be given. For $a \in A$, we write $g(a)$ instead of $g(a, 0)$. Furthermore $g(a)$ has an X-component and a Y^I-component and we write

$$g(a) = (g_1(a), g_2(a)) \in P_f \subset X \times Y^I.$$

Note that since $g(a)$ is in the pullback, $g_1(a)$ maps via f to the starting point of the path $g_2(a)$ and the square above commutes, so the endpoint of the path $g_2(a)$ is the starting point of the path $H(a, -)$. Here is a picture of $g(a)$ and $H(a, -)$.

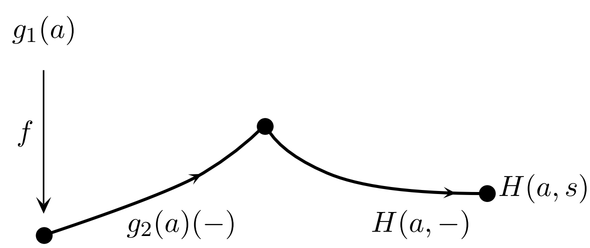


The lift $\widetilde{H}$ will have two components. The X-component will be constant in s,

$$\widetilde{H}_1(a, s) = g_1(a)$$

The Y^I-component of the lift will be a path running along the path $g_2(a)$ and then part way along $H(a, -)$, ending at $H(a, s)$.

Here is a picture of $\widetilde{H}(a, s)$.



A formula is given by

$$\widetilde{H}(a,s) = (g_1(a), \widetilde{H}_2(a,s)(-)) \in P_f \subset X \times Y^I,$$

where

$$\widetilde{H_2}(a,s)(t) = \begin{cases} g_2(a)((1+s)t) & \text{if } 0 \leq t \leq 1/(1+s), \\ H(a,((1+s)t-1) & \text{if } 1/(1+s) \leq t \leq 1. \end{cases}$$

We leave it to the reader to check $\widetilde{H}$ is continuous and that it is a lift of H extending the map g. Thus we have shown the mapping path fibration is a fibration.

3. Finally suppose that $f : X \to Y$ is itself a fibration. In the proof of 1. we showed that

$$h : X \to P_f, \qquad h(x) = (x, \mathrm{const}_{f(x)})$$

and

$$p_1 : P_f \to X \qquad p_1(x,\alpha) = x$$

are homotopy inverses. Note h is a map of fibrations (covering the identity), but p_1 is not, since $f \circ p_1(x,\alpha)$ is the starting point of α and $p(x,\alpha)$ is the endpoint of α.

Let $\gamma : P_f \times I \to Y$ be the map $\gamma(x,\alpha,t) = \alpha(t)$. Since f is a fibration, the homotopy lifting problem

$$\begin{tikzcd} P_f \times \{0\} \arrow[r, "p_1"] \arrow[d] & X \arrow[d, "f"] \\ P_f \times I \arrow[r, "\gamma"'] \arrow[ur, dotted, "\tilde{\gamma}"] & Y \end{tikzcd}$$

has a solution. Define $g : P_f \to X$ by $g(x,\alpha) = \tilde{\gamma}(x,\alpha,1)$. Then the diagrams

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & P_f \arrow[dl] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, "f"'] & & P_f \arrow[ll, "g"'] \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

commute.

Thus h and g are maps of fibrations, and in fact homotopy inverses since g is homotopic to p_1. But this is not enough.

To finish the proof, we need to show that $g \circ h$ is homotopic to Id_X by a *vertical homotopy* (i.e. a homotopy over the identity $\mathrm{Id}_Y : Y \to Y$) and $h \circ g$ is homotopic to Id_{P_f} by a vertical homotopy.

Let $F : X \times I \to X$ be the map

$$F(x,t) = \tilde{\gamma}(x,\mathrm{const}_{f(x)},t).$$

Then

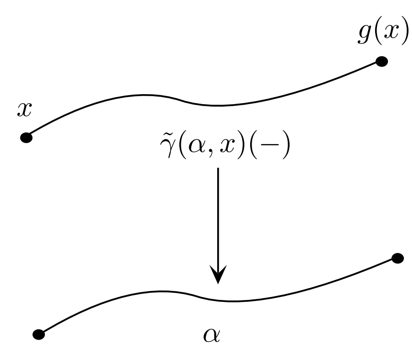
1. $F(x,0) = \tilde{\gamma}(x,\mathrm{const}_{f(x)},0) = p_1(x,\mathrm{const}_{f(x)}) = x$, and
2. $F(x,1) = \tilde{\gamma}(x,\mathrm{const}_{f(x)},1) = g \circ h(x)$.

Hence F is a homotopy from Id_X to $g \circ h$. Moreover,

$$f(F(x,t)) = f(\tilde{\gamma}(x,\mathrm{const}_{f(x)},t)) = \gamma(x,\mathrm{const}_{f(x)},t) = f(x)$$

so F is a vertical homotopy.

Here is a picture of $\tilde{\gamma}$



The vertical homotopy from Id_{P_f} to $h \circ g$ is given by contracting along paths to their endpoints. Explicitly $H : P_f \times I \to P_f$ is

$$H(x, \alpha, s) = (\tilde{\gamma}(x, \alpha, s), (t \mapsto \alpha(s + t - st))).$$

$\square$

Given a map $f : X \to Y$, it is common to be sloppy and say "F is the fiber of f", or "$F \hookrightarrow X \to Y$ is a fibration" to mean that after replacing X by the homotopy equivalent space P_f and the map f by the fibration $P_f \to Y$, the fiber is a space of the homotopy type of F.

6.7. Cofibrations

Definition 6.19. A map $i : A \to X$ is called a *cofibration*, or *satisfies the homotopy extension property (HEP)*, if the following diagram has a solution for any space Y.

$$\begin{tikzcd} A \times \{0\} \arrow[rr] \arrow[dd, "i"'] & & A \times I \arrow[dl] \arrow[dd, "i\times\mathrm{Id}"] \\ & Y & \\ X \times \{0\} \arrow[ur] \arrow[rr] & & X \times I \arrow[ul, dotted] \end{tikzcd}$$

Cofibration is a "dual" notion to fibration, using the adjointness of the functors $- \times I$ and $-^I$, and reversing the arrows. To see this, note that since a map $A \times I \to B$ is the same as a map $A \to B^I$, the diagram defining a fibration $f : X \to Y$ can be written

$$\begin{tikzcd} X & X^I \arrow[l, "\text{eval. at } 0"'] \arrow[d, "f"] \\ Z \arrow[u] \arrow[r] \arrow[ur, dotted] & Y^I. \end{tikzcd}$$

The diagram defining a cofibration $f : Y \to X$ can be written as

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d] & X \times I \arrow[dl, dotted] \\ Z & Y \times I. \arrow[l] \arrow[u, "f\times I"'] \end{tikzcd}$$

For "reasonable" spaces, any cofibration $i : A \to X$ can be shown to be an embedding whose image is closed in X. We will only deal with cofibrations given by a pair (X, A) with A a closed subspace. In that case one usually says that $A \hookrightarrow X$ is a cofibration if the problem

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "f\cup h"] \arrow[d, "i"'] & Y \\ X \times I \arrow[ur, dotted] & \end{tikzcd}$$

as a solution for all spaces Y, maps $f : X \to Y$ and homotopies $h : A \times ItoY$ extending $f|_A$. Hence the name homotopy extension property.

Definition 6.20. Let X be compactly generated, $A \subset X$ a subspace. Then (X,A) is called an *NDR–pair* (NDR stands for "neighborhood deformation retract") if there exist continuous maps $u : X \to I$ and $h : X \times I \to X$ so that:

1. $A = u^{-1}(0)$,
2. $h(-,0) = \mathrm{Id}_X$,
3. $h(a,t) = a$ for all $t \in I,\ a \in A$, and
4. $h(x,1) \in A$ for all $x \in X$ such that $u(x) < 1$.

In particular the neighborhood $U = \{x \in X | u(x) < 1\}$ of A deformation retracts to A.

Definition 6.21. A pair (X,A) is called a a *DR–pair* (DR stands for "deformation retract") if 1,2,3 hold, but also

$4'$ $h(x,1) \in A$ for all $x \in X$.

(This is slightly stronger than the usual definition of deformation retracts, because of the requirement that there exists a function $u : X \to I$ such that $u^{-1}(0) = A$.)

Theorem 6.22 (Steenrod)**.** *Equivalent are:*

1. (X,A) *is an NDR pair.*
2. $(X \times I, X \times 0 \cup A \times I)$ *is a DR pair.*
3. $X \times 0 \cup A \times I$ *is a retract of* $X \times I$.
4. $i : A \hookrightarrow X$ *is a cofibration.*

For a complete proof see Steenrod's paper [**38**].

Proof of some implications.

$(4 \Rightarrow 3)$ Let $Y = X \times 0 \cup A \times I$. Then the solution of

$$\begin{tikzcd} X \times \{0\} \,\cup\, A \times I \arrow[r, "\mathrm{Id}"] \arrow[d] & X \times \{0\} \,\cup\, A \times I \\ X \times I \arrow[ur, dotted, "r"'] & \end{tikzcd}$$

is a retraction of $X \times I$ to $X \times 0 \cup A \times I$.

$(3 \Rightarrow 4)$ The problem

$$\begin{tikzcd} X \times \{0\} \,\cup\, A \times I \arrow[r, "f"] \arrow[d] & Y \\ X \times I \arrow[ur, dotted] & \end{tikzcd}$$

has a solution $f \circ r$, where $r : X \times I \to X \times \{0\} \cup A \times I$ is the retraction.

$(1 \Rightarrow 3)$ (This implication says that NDR pairs satisfy the homotopy extension property. This is the most important property of NDR pairs.)

The map $R : X \times I \to X \times \{0\} \cup A \times I$ given by

$$R(x,t) = \begin{cases} (x,t) & \text{if } x \in A \text{ or } t = 0, \\ (h(x,1), t - u(x)) & \text{if } t \geq u(x) \text{ and } t > 0, \text{ and} \\ (h(x, \frac{t}{u(x)}), 0) & \text{if } u(x) \geq t \text{ and } u(x) > 0 \end{cases}$$

is a well-defined and continuous retraction.

□

The next result should remind you of the result that fiber bundles over paracompact spaces are fibrations.

Theorem 6.23. *If X is a CW-complex, and $A \subset X$ a subcomplex, then (X,A) is a NDR pair.*

Sketch of proof. The complex X is obtained from A by adding cells. Use a collar $S^{n-1} \times [0,1] \subset D^n$ given by $(\vec{v}, t) \mapsto (1 - \frac{t}{2})\vec{v}$ to define u and h cell-by-cell. □

Exercise 91. If (X, A) and (Y, B) are cofibrations, so is their product

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y).$$

We next establish that a pushout of a cofibration is a cofibration; this is dual to the fact that pullback of a fibration is a fibration. The word dual here is used in the sense of reversing arrows.

Definition 6.24. A *pushout of maps $f : A \to B$ and $g : A \to C$* is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

which is initial among all such commutative diagrams, i.e. any problem of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[ddr] & \\ C \arrow[r] \arrow[drr] & D \arrow[dr, dotted] & \\ & & E \end{tikzcd}$$

has a unique solution.

Pushouts are unique up to homeomorphism; this is proved using an "initial object" argument. Sometimes we just say D is the pushout, and sometimes we write $D = f_*C$, the pushout of g along f.

Pushouts always exist. They are constructed as follows.

When A is empty the pushout is the *disjoint union* $B \amalg C$. A concrete realization is given by choosing base points $b_0 \in B$ and $c_0 \in C$ and setting

$$B \amalg C = \{(b, c_0, 0) \in B \times C \times I \mid b \in B\} \ \cup \ \{(b_0, c, 1) \in B \times C \times I \mid c \in C\}.$$

In general, a concrete realization for the pushout of $f : A \to B$ and $g : A \to C$ is

$$\frac{B \amalg C}{f(a) \sim g(a)}.$$

Note that this is a quotient of a sum, just like the pushout in the category of abelian groups.

Theorem 6.25. *If $g : A \to C$ is a cofibration and*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & f_*C \end{tikzcd}$$

is a pushout diagram then $B \to f_*C$ is a cofibration. □

The proof is obtained by reversing the arrows in the dual argument for fibrations. We leave it as an exercise.

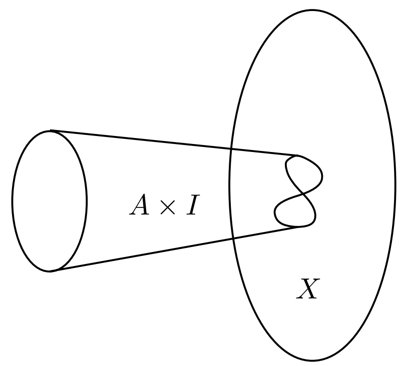
Exercise 92. Prove Theorem 6.25.

6.8. Replacing a map by a cofibration

Let $f : A \to X$ be a continuous map. We will replace X by a homotopy equivalent space M_f and obtain a map $A \to M_f$ which is a cofibration. In short, every map is equivalent to a cofibration. If f is a cofibration to begin with, then the construction gives a homotopy equivalent cofibration relative to A.

Definition 6.26. The *mapping cylinder* of a map $f : A \to X$ is the space

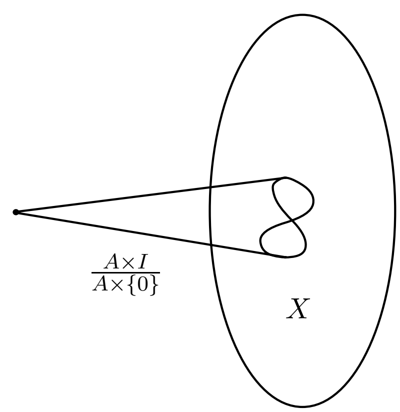
$$M_f = \frac{(A \times I) \amalg X}{(a, 1) \sim f(a)}.$$



M_f

The *mapping cone* of $f : A \to X$ is

$$C_f = \frac{M_f}{A \times \{0\}}.$$



C_f

Note that the mapping cylinder M_f can also be defined as the pushout of

$$\begin{tikzcd} A \times \{1\} \arrow[r] \arrow[d] & X \times \{1\} \\ A \times I & \end{tikzcd}$$

This shows the analogue with the mapping path fibration P_f more clearly. Sometimes P_f is called the *mapping cocylinder* by those susceptible to categorical terminology.

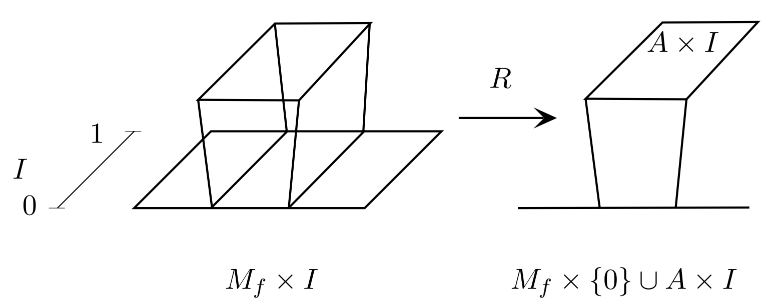
The "dual" result to Theorem 6.18 is the following.

Theorem 6.27. *Let $f : A \to X$ be a map. Let $i : A \to M_f$ be the inclusion $i(a) = [a, 0]$.*

1. *There exists a homotopy equivalence $h : M_f \to X$ so that the diagram*
$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "i"] & \\ X & & M_f \arrow[ll, "h"] \end{tikzcd}$$
commutes.
2. *The inclusion $i : A \to M_f$ is a cofibration.*
3. *If $f : A \to X$ is a cofibration, then h is a homotopy equivalence rel A, in particular h induces a homotopy equivalence of the cofibers $C_f \to X/f(A)$.*

Proof. 1. Let $h : M_f \to X$ be the map
$$h[a, s] = f(a), \qquad h[x] = x.$$
Then $f = h \circ i$ so the diagram commutes. The homotopy inverse of h is the inclusion $j : X \to M_f$. In fact, $h \circ j = \mathrm{Id}_X$, and the homotopy from Id_{M_f} to $j \circ h$ squashes the mapping cylinder onto X and is given by
$$F([a, s], t) = [a, s + t - st]$$
$$F([x], t) = [x].$$

2. By the implication $(3 \Rightarrow 4)$ from Steenrod's theorem (Theorem 6.22), we need to construct a retraction
$$R : M_f \times I \to M_f \times 0 \cup A \times I$$



Let
$$r : I \times I \to I \times 0 \cup 0 \times I$$

be a retraction so that $r(1 \times I) = \{(1,0)\}$. (First retract the square onto 3 sides and then contract a side to a point.) Define $R([a,s],t) = [a, r(s,t)]$ and $R([x],t) = ([x],0)$. Thus $i : A \to M_f$ is a cofibration.

3. If $f : A \hookrightarrow X$ is a cofibration, by Steenrod's theorem there is a retraction

$$r : X \times I \to X \times 1 \cup f(A) \times I$$

and an obvious homeomorphism

$$q : X \times 1 \cup f(A) \times I \to M_f.$$

Define $g : X \to M_f$ by $g(x) = q(r(x,0))$. We will show that g and h are homotopy inverses rel A (recall $h[a,s] = f(a)$ and $h[x] = x$).

Define the homotopy

$$H : X \times I \to X$$

as $H = h \circ r$. Then $H(x,0) = h \circ g(x)$, $H(x,1) = x$, and $H(f(a),t) = f(a)$. Define the homotopy

$$F : M_f \times I \to M_f$$

by $F([x],t) = q(r(x,t))$ and $F([a,s],t) = q(r(f(a),st))$. Then $F(x,0) = g \circ h(x)$, $H(-,1) = \mathrm{Id}_{M_f}$, and $F(i(a),t) = i(a)$. The reader is encouraged to verify these formulae, or to draw the motivating pictures. $\square$

6.9. Sets of homotopy classes of maps

We introduce the following notation. If X, Y are spaces, then $[X,Y]$ denotes the set of homotopy classes of maps from X to Y, i.e.

$$[X,Y] = \mathrm{Map}(X,Y)/\sim$$

where $f \sim g$ if f is homotopic to g.

Notice that if Y is path-connected, then the set $[X,Y]$ contains a distinguished class of maps, namely the unique class containing all the constant maps. We will use this as a base point for $[X,Y]$ if one is needed.

If X has a base point x_0, and Y has a base point y_0, let $[X,Y]_0$ denote the *based homotopy classes of based maps*, where a *based map* is a map $f : (X,x_0) \to (Y,y_0)$. Then $[X,Y]_0$ has a distinguished class, namely the class of the constant map at y_0. (In the based context, it is not necessary to assume Y is path-connected to have this distinguished class.) Given a map $f : X \to Y$ let $[f]$ denote its homotopy class in $[X,Y]$ or $[X,Y]_0$. Notice that if X and Y are based spaces there is a forgetful map $[X,Y]_0 \to [X,Y]$. This map need not be injective or surjective.

The notion of an exact sequence of sets is a useful generalization of the corresponding concept for groups.

Definition 6.28. A sequence of functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of *sets* (not spaces or groups) *with base points* is called *exact at* B if

$$f(A) = g^{-1}(c_0)$$

where c_0 is the base point of C.

All that was necessary here was that C be based. Notice that if A, B, C are groups, with basepoints the identity element, and f, g homomorphisms, then $A \to B \to C$ is exact *as a sequence of sets* if and only if it is exact as a sequence of groups.

The following two theorems form the cornerstone of constructions of exact sequences in algebraic topology.

Theorem 6.29 (basic property of fibrations)**.** *Let $p : E \to B$ be a fibration, with fiber $F = p^{-1}(b_0)$ and B path-connected. Let Y be any space. Then the sequence of sets*

$$[Y,F] \xrightarrow{i_*} [Y,E] \xrightarrow{p_*} [Y,B]$$

is exact.

Proof. Clearly $p_*(i_*[g]) = 0$.

Suppose $f : Y \to E$ so that $p_*[f] = [\text{const}]$, i.e. $p \circ f : Y \to B$ is null homotopic. Let $G : Y \times I \to B$ be a null homotopy, and then let $H : Y \times I \to E$ be a solution to the lifting problem

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "G"'] \arrow[ur, dotted, "H"] & B
\end{tikzcd}$$

Since $p \circ H(y,1) = G(y,1) = b_0$, $H(y,1) \in F = p^{-1}(b_0)$. Thus f is homotopic into the fiber, so $[f] = i_*[H(-,1)]$. □

Theorem 6.30 (basic property of cofibrations)**.** *Let $i : A \hookrightarrow X$ be a cofibration, with cofiber X/A. Let $q : X \to X/A$ denote the quotient map. Let Y be any path-connected space. Then the sequence of sets*

$$[X/A,Y] \xrightarrow{q^*} [X,Y] \xrightarrow{i^*} [A,Y]$$

is exact.

Proof. Clearly $i^*(q^*([g])) = [g \circ q \circ i] = [\text{const}]$.

Suppose $f : X \to Y$ is a map and suppose that $f_{|A} : A \to Y$ is nullhomotopic. Let $h : A \times I \to Y$ be a null homotopy. The solution F to the problem

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "f \cup h"] \arrow[d, "i"'] & Y \\ X \times I \arrow[ur, dotted, "F"'] & \end{tikzcd}$$

defines a map $f' = F(-,1)$ homotopic to f whose restriction to A is constant, i.e. $f'(A) = y_0$. Therefore the diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "q"'] & Y \\ X/A \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

can be completed, by the definition of quotient topology. Thus $[f] = [f'] = q^*[g]$. □

6.10. Adjoint of loops and suspension; smash products

Definition 6.31. Define $\mathcal{K}_*$ to be the category of compactly generated spaces with a *non-degenerate base point*, i.e. (X,x_0) is an object of $\mathcal{K}_*$ if the inclusion $\{x_0\} \subset X$ is a cofibration. The morphisms in $\mathcal{K}_*$ are the base point preserving continuous maps.

Exercise 93. Prove the base-point versions of the previous two theorems:

1. If $F \hookrightarrow E \to B$ is a base point preserving fibration, then for any $Y \in \mathcal{K}_*$
$$[Y,F]_0 \to [Y,E]_0 \to [Y,B]_0$$
is exact.
2. If $A \hookrightarrow X \to X/A$ is a base point preserving cofibration, then for any $Y \in \mathcal{K}_*$
$$[X/A,Y]_0 \to [X,Y]_0 \to [A,Y]_0$$
is exact.

Most exact sequences in algebraic topology can be derived from Theorems 6.29, 6.30, and Exercise 93. We will soon use this exercise to establish exact sequences of homotopy groups. To do so, we need to be careful about base points and adjoints. Recall that if (X, x_0) and (Y, y_0) are based spaces, then $\mathrm{Map}(X,Y)_0$ is the set of maps of pairs $(X,x_0) \to (Y,y_0)$ with the compactly generated topology.

Definition 6.32. The *smash product* of based spaces is

$$X \wedge Y = \frac{X \times Y}{X \vee Y} = \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \cup Y}.$$

Note that the smash product $X \wedge Y$ is a based space. Contrary to popular belief, the smash product is *not* the product in the category $\mathcal{K}_*$, although the *wedge product*

$$X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subset X \times Y$$

is the sum in $\mathcal{K}_*$. The smash product is the adjoint of the based mapping space. The following theorem follows from the unbased version of the adjoint theorem (Theorem 6.5), upon restricting to based maps.

Theorem 6.33 (adjoint theorem). *There is a (natural) homeomorphism*

$$\mathrm{Map}(X \wedge Y, Z)_0 \cong \mathrm{Map}(X, \mathrm{Map}(Y,Z)_0)_0$$

$\square$

Definition 6.34. The *(reduced) suspension* of a based space (X, x_0) is $SX = S^1 \wedge X$. The *(reduced) cone* is $CX = I \wedge X$. Here the circle is based by $1 \in S^1 \subset \mathbf{C}$ and the interval by $0 \in I$.

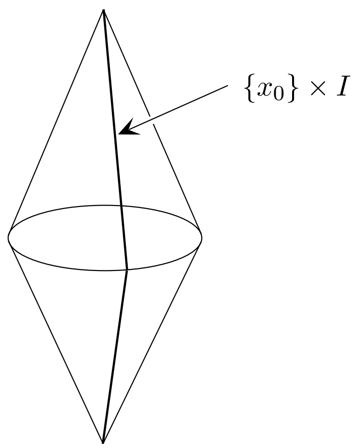
Using the usual identification $I/\{0,1\} = S^1$ via $t \mapsto e^{2\pi i t}$, one sees

$$SX = \frac{X \times I}{X \times \{0,1\} \cup \{x_0\} \times I}$$

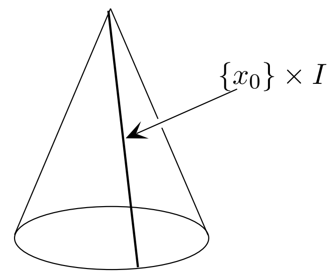
In other words, if ΣX is the unreduced suspension and $\mathrm{cone}(X)$ is the unreduced cone ($= \Sigma X / X \times \{0\}$), then there are quotient maps

$$\Sigma X \to SX \qquad \mathrm{cone}(X) \to CX$$

given by identifying $\{x_0\} \times I$ shaded in the following figure.



ΣX



$\text{cone}(X)$

Notice that taking reduced suspensions and reduced cones is functorial. Reduced suspensions and cones are more useful than the unreduced variety since they have canonical base points and satisfy adjoint properties. Nonetheless, it is reassuring to connect them with the more familiar unreduced versions.

Exercise 94. If $X \in \mathcal{K}_*$ (i.e. the inclusion $\{x_0\} \to X$ is a cofibration), then the quotient maps $\Sigma X \to SX$ and $\text{cone}(X) \to CX$ are homotopy equivalences.

Proposition 6.35. *The reduced suspension SS^n is homeomorphic to S^{n+1} and the reduced cone CS^n is homeomorphic to D^{n+1}.* □

Exercise 95. Prove Proposition 6.35. This shows in a special case that the smash product is associative. Prove associativity of the smash product in general.

Corollary 6.36. *$S^i \wedge S^j$ is homeomorphic to S^{i+j}.* □

We defined loop spaces by $\Omega_{x_0} X = \text{Map}(I, \{0,1\}; X, \{x_0\})$, but by using the identification of the circle as a quotient space of the interval, one sees

$$\Omega_{x_0} X = \text{Map}(S^1, X)_0$$

Then a special case of Theorem 6.33 shows the following.

Theorem 6.37 (loops and suspension are adjoints)**.** *The spaces*

$$\text{Map}(SX, Y)_0$$

and

$$\text{Map}(X, \Omega Y)_0$$

are naturally homeomorphic. □

6.11. Fibration and cofibration sequences

We will see eventually that the homotopy type of a fiber of a fibration measures how far the fibration is from being a homotopy equivalence. (For example, if the fiber is contractible then the fibration is a homotopy equivalence.) More generally given a map $f : X \to Y$, one can turn it into a fibration $P_f \to Y$ as above; the fiber of this fibration measures how far f is from a homotopy equivalence.

After turning $f : X \to Y$ into a fibration $P_f \to Y$ one then has an inclusion of the fiber $F \subset P_f$. Why not turn this into a fibration and see what happens? Now take the fiber of the resulting fibration and continue the process ...

Similar comments apply to cofibrations. Theorem 6.39 below identifies the resulting iterated fibers and cofibers. We first introduce some terminology.

Definition 6.38. If $f : X \to Y$ is a map, the *homotopy fiber of* f is the fiber of the fibration obtained by turning f into a fibrations. The homotopy fiber is a space, well-defined up to homotopy equivalence. Usually one is lazy and just calls this the *fiber of* f.

Similarly, the *homotopy cofiber of* $f : X \to Y$ is the mapping cone C_f, the cofiber of $X \to M_f$.

Theorem 6.39.

1. *Let $F \hookrightarrow E \to B$ be a fibration. Let Z be the homotopy fiber of $F \hookrightarrow E$, so $Z \to F \to E$ is a fibration (up to homotopy). Then Z is homotopy equivalent to the loop space ΩB.*
2. *Let $A \hookrightarrow X \to X/A$ be a cofibration sequence. Let W be the homotopy cofiber of $X \to X/A$, so that $X \to X/A \to W$ is a cofibration (up to homotopy). Then W is homotopy equivalent to the (unreduced) suspension ΣA.*

Proof. 1. Let $f : E \to B$ be a fibration with fiber $F = f^{-1}(b_0)$. Choose a base point $e_0 \in F$. In Section 6.6 we constructed a fibration $p : P_f \to B$ with

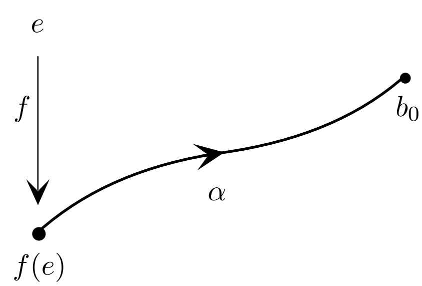
$$P_f = \{(e,\alpha) \in E \times B^I | f(e) = \alpha(0)\}$$

and $p(e,\alpha) = \alpha(1)$, and such that the map $h : E \to P_f$ given by $h(e) = (e, \mathrm{const}_{f(e)})$ is a fiber homotopy equivalence.

Let $(P_f)_0 = p^{-1}(b_0)$, so $(P_f)_0 \hookrightarrow P_f \xrightarrow{p} B$ is a fibration equivalent to $F \hookrightarrow E \xrightarrow{f} B$.

Define $\pi : (P_f)_0 \to E$ by $\pi(e,\alpha) = e$. Notice that

$$(P_f)_0 = \{(e,\alpha) | f(e) = \alpha(0),\ \alpha(1) = b_0\}.$$

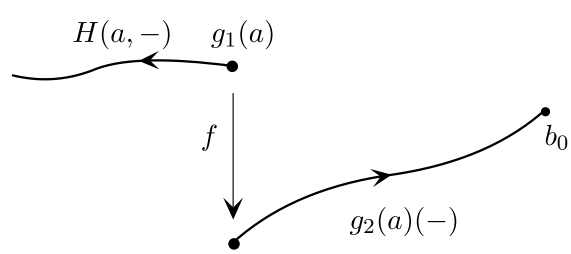


Claim. $\pi : (P_f)_0 \to E$ is a fibration with fiber $\Omega_{b_0} B$.

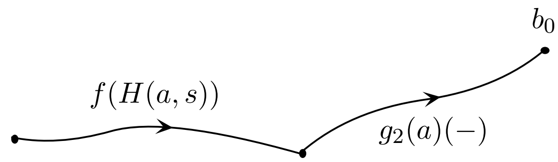
Proof of claim. Clearly $\pi^{-1}(e_0) = \{(e_0,\alpha) | \alpha(0) = \alpha(1) = b_0\}$ is homeomorphic to the loop space, so we just need to show π is a fibration. Given the problem

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "g"] \arrow[d] & (P_f)_0 \arrow[d, "\pi"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & E
\end{tikzcd}$$

the picture is



Hence we can set $\widetilde{H}(a,s) = (H(a,s), \widetilde{H}_2(a,s))$ where $\widetilde{H}_2(a,s))(-)$ has the picture



and is defined by

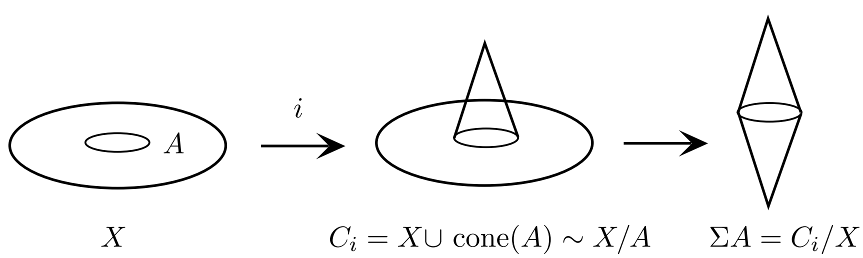
$$\widetilde{H}_2(a,s))(t) = \begin{cases} f(H(y, -(1+s)t+s)) & \text{if } 0 \le t \le s/(s+1), \\ g_2(a)((s+1)t - s & \text{if } \ s/(s+1) \le t \le 1. \end{cases}$$

The map $F \hookrightarrow (P_f)_0$ is a homotopy equivalence, since $E \to P_f$ is a fiber homotopy equivalence. Thus the diagram

$$\begin{tikzcd} F \arrow[dd, "\simeq"'] \arrow[dr] & \\ & E \\ (P_f)_0 \arrow[ur, "\pi"'] & \end{tikzcd}$$

shows that the fibration $\pi : (P_f)_0 \to E$ is obtained by turning $F \hookrightarrow E$ into a fibration, and the homotopy fiber is $\Omega_{b_0} B$.

2. The map $X \to X/A$ is equivalent to $X \hookrightarrow C_i = X \cup \text{cone}(A)$ where $i : A \hookrightarrow X$. The following picture makes clear that $C_i/X = \Sigma A$. The fact that $X \to C_i$ is a cofibration is left as an exercise.



$\square$

Exercise 96. Show that $X \hookrightarrow C_i = X \cup \text{cone}(A)$ is a cofibration.

We have introduced the notion of the loop space ΩX of a based space X as the space of paths in X which start and end at the base point. The loop space is itself a based space with base point the constant loop at the base point of X. Let $\Omega^n X$ denote the n-fold loop space of X. Similarly the reduced suspension SX of X is a based space. Let $S^n X$ denote the n-fold suspension of X.

The previous theorem can be restated in the following convenient form.

Theorem 6.40.

1. *Let $A \hookrightarrow X$ be a cofibration. Then any two consecutive maps in the sequence*

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \cdots \to \Sigma^n A \to \Sigma^n X \to \Sigma^n(X/A) \to \cdots$$

have the homotopy type of a cofibration followed by projection onto the cofiber.

1′. *Let $A \hookrightarrow X$ be a base point preserving cofibration. Then any two consecutive maps in the sequence*

$$A \to X \to X/A \to SA \to SX \to \cdots \to S^n A \to S^n X \to S^n(X/A) \to \cdots$$

have the homotopy type of a cofibration followed by projection onto the cofiber.

2. *Let $E \to B$ be a fibration with fiber F. Then any two consecutive maps in the sequence*

$$\cdots \to \Omega^n F \to \Omega^n E \to \Omega^n B \to \cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

have the homotopy type of a fibration preceded by the inclusion of its fiber.

□

To prove 1′., one must use reduced mapping cylinders and reduced cones.

6.12. Puppe sequences

Lemma 6.41. *Let X and Y be spaces in $\mathcal{K}_*$.*

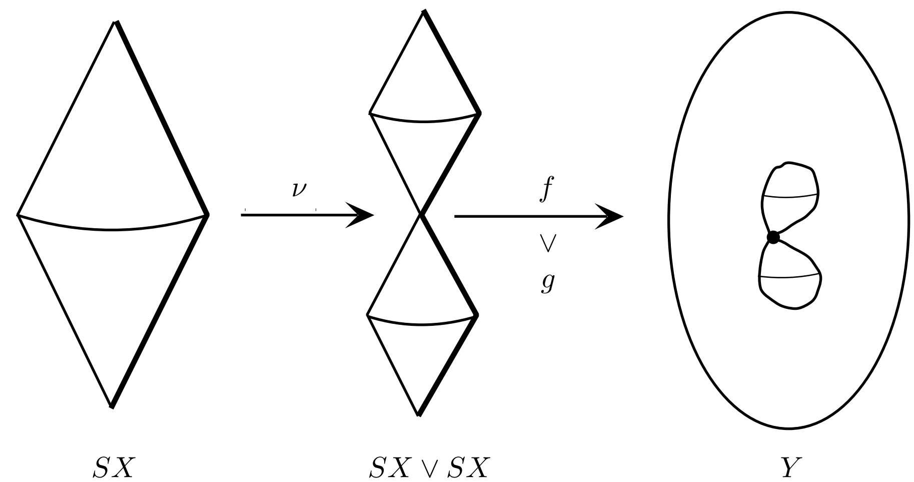
1. $[X,\Omega Y]_0 = [SX,Y]_0$ *is a group.*
2. $[X,\Omega(\Omega Y)]_0 = [SX,\Omega Y]_0 = [S^2X,Y]_0$ *is an abelian group.*

Sketch of proof. The equalities follow from Theorem 6.37, the adjointness of loops and suspension. The multiplication can be looked at in two ways: first on $[SX,Y]_0$ as coming from the map

$$\nu : SX \to SX \vee SX$$

given by collapsing out the "equator" $X \times 1/2$. Then define

$$fg \underset{\text{def}}{:=} \nu(f \vee g)$$

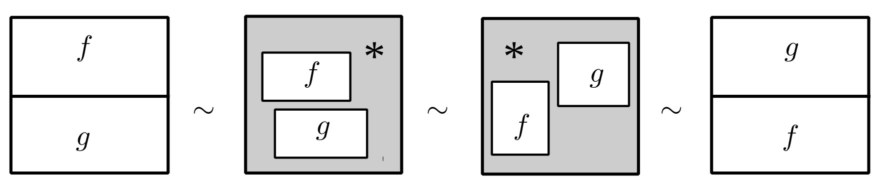


The second interpretation of multiplication is on $[X, \Omega Y]_0$ and comes from composition of loops

$$* : \Omega Y \times \Omega Y \to \Omega Y$$

with $(fg)x = f(x) * g(x)$.

The proof of 2 is obtained by meditating on the following sequence of pictures.



□

Exercise 97. Convince yourself that the two definitions of multiplication on $[X,\Omega Y]_0 = [SX,Y]_0$ are the same and that $\pi_1(Y, y_0) = [SS^0, Y]_0$.

The last lemma sits in a more general context. A loop space is a example of an H-group and a suspension is an example of a co-H-group. See [**36**] or [**43**] for precise definitions, but here is the basic idea. An H-group Z is a based space with a "multiplication" map $\mu : Z \times Z \to Z$ and an "inversion" map $\varphi : X \to X$ which satisfy the axioms of a group up to homotopy (e.g. is associative up to homotopy). For a topological group G and any space X, $\mathrm{Map}(X, G)$ is a group, similarly for an H-group Z, $[X, Z]_0$ is a group. To define a co-H-group, one reverses all the arrows in the definition of H-group, so there is a co-multiplication $\nu : W \to W \vee W$ and a co-inversion

$\psi : W \to W$. Then $[W, X]_0$ is a group. Finally, there is a formal, but occasionally very useful result. If W is a co-H-group and Z is an H-group, then the two multiplications on $[W, Z]_0$ agree and are abelian. Nifty, huh? One consequence of this is that $\pi_1(X, x_0)$ of an H-group (e.g. a topological group) is abelian.

Combining Lemma 6.41 with Theorem 6.40 and Exercise 93 yields the proof of the following fundamental theorem.

Theorem 6.42 (Puppe sequences). *Let $Y \in \mathcal{K}_*$.*

1. *If $F \to E \to B$ is a fibration, the following sequence is a long exact sequence of sets ($i \geq 0$), groups ($i \geq 1$), and abelian groups ($i \geq 2$).*
$$\cdots \to [Y, \Omega^i F]_0 \to [Y, \Omega^i E]_0 \to [Y, \Omega^i B]_0 \to \\ \cdots \to [Y, \Omega B]_0 \to [Y, F]_0 \to [Y, E]_0 \to [Y, B]_0$$
where $\Omega^i Z$ denotes the iterated loop space
$$\Omega(\Omega(\cdots(\Omega Z)\cdots).$$
2. *If (X, A) is an cofibration, the following sequence is a long exact sequence of sets ($i \geq 0$), groups ($i \geq 1$), and abelian groups ($i \geq 2$).*
$$\cdots \to [S^i(X/A), Y]_0 \to [S^i X, Y]_0 \to [S^i A, Y]_0 \to \\ \cdots \to [SA, Y]_0 \to [X/A, Y]_0 \to [X, Y]_0 \to [A, Y]_0$$

□

This theorem is used as the basic tool for constructing exact sequences in algebraic topology.

6.13. Homotopy groups

We now define the homotopy groups of a based space.

Definition 6.43. Suppose that X is a space with base point x_0. Then the *n^{th} homotopy group of X based at x_0* is the group (set if $n = 0$, abelian group if $n \geq 2$)
$$\pi_n(X, x_0) = [S^n, X]_0.$$

(We will usually only consider $X \in \mathcal{K}_*$.)

Notice that

$$\pi_n(X, x_0) = [S^n, X]_0 = [S^k \wedge S^{n-k}, X]_0 = \pi_{n-k}(\Omega^k(X)). \tag{6.2}$$

In particular,
$$\pi_n X = \pi_1(\Omega^{n-1} X).$$

There are other ways of looking at homotopy groups which are useful. For example, to get a hold of the group structure for writing down a proof, use $\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$. For the proof of the exact sequence of a pair (coming later) use $\pi_n(X, x_0) = [(D^n, S^{n-1}), (X, x_0)]$. For finding a geometric interpretation of the boundary map in the homotopy long exact sequence of a fibration given below, use

$$\pi_n(X, x_0) = [(S^{n-1} \times I, (S^{n-1} \times \partial I) \cup (* \times I)), (X, x_0)].$$

A useful observation is that the set $\pi_0(X, x_0)$ is in bijective correspondence with the path components of X. A based map $f : S^0 = \{\pm 1\} \to X$ corresponds to the path component of $f(-1)$. In general π_0 is just a based set, unless X is an H-space, e.g. a loop space or a topological group.

Also useful is the fact that $[X, Y]_0 = \pi_0(\mathrm{Map}(X, Y)_0)$, the set of path components of the function space $\mathrm{Map}(X, Y)_0$. In particular, Equation (6.2) shows that $\pi_n(X, x_0)$ is the set of path components of the n-fold loop space of X.

Homotopy groups are the most fundamental invariant of algebraic topology. For example, we will see below that a CW-complex is contractible if and only if all its homotopy groups vanish. More generally we will see that a map $f : X \to Y$ is a homotopy equivalence if and only if it induces an isomorphism on all homotopy groups. Finally, the homotopy type of a CW-complex X is determined by the homotopy groups of X together with a cohomological recipe (the k-invariants) for assembling these groups. (The homotopy groups by themselves do not usually determine the homotopy type of a space.)

Exercise 98. Show that $\pi_n(X \times Y) = \pi_n(X) \oplus \pi_n(Y)$.

As an application of the Puppe sequences (Theorem 6.42) we immediately get the extremely useful long exact sequence of homotopy groups associated to any fibration.

Corollary 6.44 (long exact sequence of a fibration)**.** *Let $F \hookrightarrow E \to B$ be a fibration. Then the sequence*

$$\begin{aligned} \cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \pi_{n-1} E \to \cdots \\ \to \pi_1 F \to \pi_1 E \to \pi_1 B \to \pi_0 F \to \pi_0 E \to \pi_0 B \end{aligned}$$

is exact. □

In Corollary 6.44, one must be careful with exactness at the right end of this sequence since $\pi_1 F$, $\pi_1 E$, and $\pi_1 B$ are non-abelian groups and $\pi_0 F$, $\pi_0 E$, and $\pi_0 B$ are merely sets.

Taking F discrete in Corollary 6.44 and using the fact that covering spaces are fibrations one concludes the following important theorem.

Theorem 6.45. *Let $\tilde{X} \to X$ be a connected covering space of a connected space X. Then the induced map*

$$\pi_n(\tilde{X}) \to \pi_n(X)$$

is injective if $n = 1$, and an isomorphism if $n > 1$. □

Exercise 99. Give a covering space proof of Theorem 6.45.

6.14. Examples of fibrations

Many examples of fibrations and fiber bundles arise naturally in mathematics. Getting a feel for this material requires getting one's hands dirty. For that reason many facts are left as exercises. We will use the following theorem from equivariant topology to conclude that certain maps are fibrations. This is a special case of Theorem 4.5.

Theorem 6.46 (Gleason)**.** *Let G be a compact Lie group acting freely on a compact manifold X. Then*

$$X \to X/G$$

is a principal fiber bundle with fiber G.

6.14.1. Hopf fibrations.

The first class of examples we give are the famous *Hopf fibrations.* These were invented by Hopf to prove that there are non–nullhomotopic maps $S^n \to S^m$ when $n > m$.

There are four Hopf fibrations (these are fiber bundles):

$$S^0 \hookrightarrow S^1 \to S^1$$

$$S^1 \hookrightarrow S^3 \to S^2$$

$$S^3 \hookrightarrow S^7 \to S^4$$

and

$$S^7 \hookrightarrow S^{15} \to S^8.$$

These are constructed by looking at the various division algebras over $\mathbf{R}$.

Let $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$, or $\mathbf{O}$ (the real numbers, complex numbers, quaternions, and octonions). Each of these has a *norm* $N : K \to \mathbf{R}_+$ so that

$$N(xy) = N(x)N(y)$$

and $N(x) > 0$ for $x \neq 0$.

More precisely,

1. If $K = \mathbf{R}$, then $N(x) = |x| = \sqrt{x\overline{x}}$ where $\overline{x} = x$,
2. If $K = \mathbf{C}$, then $N(x) = \sqrt{x\overline{x}}$ where $\overline{a + ib} = a - ib$,
3. If $K = \mathbf{H}$, then $N(x) = \sqrt{x\overline{x}}$, where $\overline{a + ib + jc + kd} = a - ib - jc - kd$,

4. The octonions are defined to be $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}$. The conjugation is defined by the rule: if $p = (a, b)$, then $\overline{p} = (\overline{a}, -b)$. Multiplication is given by the rule

$$(a, b)(c, d) = (ac - \overline{d}b, b\overline{c} + da)$$

and the norm is defined by

$$N(p) = \sqrt{p\overline{p}}.$$

Let $E_K = \{(x, y) \in K \oplus K | N(x)^2 + N(y)^2 = 1\}$. Let $G_K = \{x \in K | N(x) = 1\}$.

Exercise 100. G_K is a compact Lie group homeomorphic to S^r for $r = 0, 1, 3$. For $K = \mathbf{O}$, G_K is homeomorphic to S^7, but it is not a group; associativity fails.

Let G_K act on E_K by $g \cdot (x, y) = (gx, gy)$ (Note $N(gx)^2 + N(gy)^2 = N(x)^2 + N(y)^2$ if $N(g) = 1$.)

This action is free. This is easy to show for $K = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$, since K is associative, hence if $g(x, y) = (x, y)$, one of x or y is non-zero (since $N(x)$ and $N(y)$ are not both zero) and so if $x \neq 0$, $gx = x$ implies that $1 = xx^{-1} = (gx)x^{-1} = g(xx^{-1}) = g$. This argument does not work for $K = \mathbf{O}$ since G_K is not a group; in this case one defines an equivalence relation on E_K by $(x, y) \sim (gx, gy)$ for $g \in G_K$. The resulting quotient map $E_K \to E_k/\sim$ is a fiber bundle.

It is also easy to see that E_K consists of the unit vectors in the corresponding $\mathbf{R}^n$ and so $E_K = S^{2r+1}$ for $r = 0, 1, 3, 7$. Moreover $G_K \cong S^r$ and so the fiber bundle $G_K \hookrightarrow E_K \to E_K/G_K$ can be rewritten

$$S^r \hookrightarrow S^{2r+1} \to Y = S^{2r+1}/S^r$$

Exercise 101. Prove that Y is homeomorphic to the $(r+1)$-sphere S^{r+1} in the 4 cases. In fact, prove that the quotient map $S^{2r+1} \to Y$ can be written in the form $f : S^{2r+1} \to S^{r+1}$ where

$$f(z_1, z_2) = (2\overline{z}_1 z_2, N(z_1)^2 - N(z_2)^2).$$

Using these fibrations and the long exact sequence of a fibration (Corollary 6.44) one obtains exact sequences

$$\cdots \to \pi_n S^1 \to \pi_n S^3 \to \pi_n S^2 \to \pi_{n-1} S^1 \to \cdots$$

$$\cdots \to \pi_n S^3 \to \pi_n S^7 \to \pi_n S^4 \to \pi_{n-1} S^3 \to \cdots$$

$$\cdots \to \pi_n S^7 \to \pi_n S^{15} \to \pi_n S^8 \to \pi_{n-1} S^7 \to \cdots$$

Since $\pi_n S^1 = 0$ for $n > 1$ (the universal cover of S^1 is contractible and so this follows from Theorem 6.45), it follows from the first sequence that

$\pi_n S^3 = \pi_n S^2$ for $n > 2$. The Hopf degree Theorem (Corollary 6.67 and a project for Chapter 3) implies that $\pi_n S^n = \mathbf{Z}$. In particular,

$$\pi_3 S^2 = \mathbf{Z}.$$

This is our second non-trivial calculation of $\pi_m S^n$ (the first being $\pi_n S^n = \mathbf{Z}$).

The quickest way to to obtain information from the other sequences is to use the cellular approximation theorem. This is an analogue of the simplicial approximation theorem. Its proof is one of the projects for Chapter 1.

Theorem 6.47 (cellular approximation theorem)**.** *Let (X, A) and (Y, B) be relative CW-complexes, and let $f : (X, A) \to (Y, B)$ be a continuous map. Then f is homotopic rel A to a cellular map.* □

Applying this theorem with $(X, A) = (S^n, x_0)$ and $(Y, B) = (S^m, y_0)$ one concludes that

$$\pi_n S^m = 0 \text{ if } \ n < m.$$

Returning to the other exact sequences, it follows from the cellular approximation theorem that $\pi_n S^4 = \pi_{n-1} S^3$ for $n \leq 6$ (since $\pi_n(S^7) = 0$ for $n \leq 6$), and that $\pi_n S^8 = \pi_{n-1} S^7$ for $n \leq 14$. We will eventually be able to say more.

6.14.2. Projective spaces. The Hopf fibrations can be generalized by taking G_K acting on K^n for $n > 2$ at least for $K = \mathbf{R}, \mathbf{C}$, and $\mathbf{H}$.

For $K = \mathbf{R}$, $G_K = \mathbf{Z}/2$ acts on S^n with quotient real projective space $\mathbf{R}P^n$. The quotient map $S^n \to \mathbf{R}P^n$ is a covering space, and in particular a fibration.

Let S^1 act on

$$S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbf{C}^n \mid \Sigma |z_i|^2 = 1\}$$

by

$$t(z_1, \cdots, z_n) = (tz_1, \cdots, tz_n)$$

if $t \in S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$.

Exercise 102. Prove that S^1 acts freely.

The orbit space is denoted by $\mathbf{C}P^{n-1}$ and called *complex projective space.* The projection $S^{2n-1} \to \mathbf{C}P^{n-1}$ is a fibration with fiber S^1. (Can you prove directly that this is a fiber bundle?) In fact, if one uses the map $p : S^{2n-1} \to \mathbf{C}P^{n-1}$ to adjoin a $2n$-cell, one obtains $\mathbf{C}P^n$. Thus complex projective space is a CW-complex.

Notice that $\mathbf{C}P^n$ is a subcomplex of $\mathbf{C}P^{n+1}$, and in fact $\mathbf{C}P^{n+1}$ is obtained from $\mathbf{C}P^n$ by adding a single $2n+2$-cell. One defines infinite complex projective space $\mathbf{C}P^\infty$ to be the union of the $\mathbf{C}P^n$, with the CW-topology.

Exercise 103. Using the long exact sequence for a fibration, show that $\mathbf{C}P^\infty$ is an Eilenberg–MacLane space of type $K(\mathbf{Z}, 2)$, i.e. a CW-complex with π_2 the only non-zero homotopy group and $\pi_2 \cong \mathbf{Z}$.

Similarly, there is a fibration

$$S^3 \hookrightarrow S^{4n-1} \to \mathbf{H}P^{n-1}$$

using quaternions in the previous construction. The space $\mathbf{H}P^{n-1}$ is called *quaternionic projective space*.

Exercise 104.

1. Calculate the cellular chain complexes for $\mathbf{C}P^k$ and $\mathbf{H}P^k$.
2. Compute the *ring* structure of $H^*(\mathbf{C}P^k; \mathbf{Z})$ and $H^*(\mathbf{H}P^k; \mathbf{Z})$ using Poincaré duality.
3. Examine whether $\mathbf{O}P^k$ can be defined this way, for $k > 1$.
4. Show these reduce to Hopf fibrations for $k = 1$.

6.14.3. More general homogeneous spaces and fibrations.

Definition 6.48.

1. The *Stiefel manifold* $V_k(\mathbf{R}^n)$ is the space of orthonormal k-frames in $\mathbf{R}^n$:
$$V_k(\mathbf{R}^n) = \{(v_1, v_2, \ldots, v_k) \in (\mathbf{R}^n)^k \mid v_i \cdot v_j = \delta_{ij}\}$$
given the topology as a subspace of $(\mathbf{R}^n)^k = \mathbf{R}^{nk}$.
2. The *Grassmann manifold* or *grassmannian* $G_k(\mathbf{R}^n)$ is the space of k-dimensional subspaces (a.k.a. k-planes) in $\mathbf{R}^n$. It is given the quotient topology using the surjection $V_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n)$ taking a k-frame to the k-plane it spans.

Let G be a compact Lie group. Let $H \subset G$ be a closed subgroup (and hence a Lie group itself). The quotient G/H is called a *homogeneous space*. The (group) quotient map $G \to G/H$ is a principal H-bundle since H acts freely on G by right translation. If H has a closed subgroup K, then H acts on the homogeneous space H/K. Changing the fiber of the above bundle results in a fiber bundle $G/K \to G/H$ with fiber H/K.

For example, if $G = O(n)$ and $H = O(k) \times O(n-k)$ with $H \hookrightarrow G$ via

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

let $K \subset O(n)$ be $O(n-k)$, with

$$A \mapsto \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}.$$

Exercise 105. Identify G/H with the grassmannian and G/K with the Stiefel manifold. Conclude that the map taking a frame to the plane it spans defines a principal $O(k)$ bundle $V_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n)$.

Let

$$\gamma_k(\mathbf{R}^n) = \{(p, V) \in \mathbf{R}^n \times G_k(\mathbf{R}^n) \mid p \text{ is a point in the } k\text{-plane } V\}.$$

There is a natural map $\gamma_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n)$ given by projection on the second coordinate. The fiber bundle so defined is a vector bundle with fiber $\mathbf{R}^k$ (a *k-plane bundle*)

$$\mathbf{R}^k \hookrightarrow \gamma_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n).$$

It is called the canonical (or tautological) vector bundle over the grassmannian.

Exercise 106. Identify the canonical bundle with the bundle obtained from the principal $O(k)$ bundle $V_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n)$ by changing the fiber to $\mathbf{R}^k$.

Exercise 107. Show there are fibrations

$$O(n-k) \hookrightarrow O(n) \to V_k(\mathbf{R}^n)$$
$$O(n-1) \hookrightarrow O(n) \to S^{n-1}$$

taking a matrix to its last k columns. Deduce that

$$\pi_i(O(n-1)) \cong \pi_i(O(n)) \quad \text{for} \quad i < n-2, \tag{6.3}$$

and

$$\pi_i(V_k(\mathbf{R}^n)) = 0 \quad \text{for} \quad i < n-k-1.$$

The isomorphism of Equation (6.3) is an example of "stability" in algebraic topology. In this case it leads to the following construction. Consider the infinite orthogonal group

$$O = \lim_{n\to\infty} O(n) = \bigcup_{n=1}^{\infty} O(n),$$

where $O(n) \subset O(n+1)$ is given by the continuous monomorphism

$$A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Topologize O as the expanding union of the $O(n)$. Then any compact subset of O is contained in $O(n)$ for some n, hence $\pi_i O = \lim_{n\to\infty} \pi_i(O(n)) = \pi_i(O(n))$ for any $n > i+2$.

A famous theorem of Bott says:

Theorem 6.49 (Bott periodicity)**.**

$$\pi_k O \cong \pi_{k+8} O \quad \textit{for } k \in \mathbf{Z}_+.$$

Moreover the homotopy groups of O are computed to be

$k \pmod 8$	0	1	2	3	4	5	6	7
$\pi_k O$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$

$\square$

An element of $\pi_k O$ is given by an element of $\pi_k(O(n))$, for some n, which by clutching (see Section 4.3.3) corresponds to a bundle over S^{k+1} with structure group $O(n)$. (Alternatively, one may use that $\pi_{k+1}(BO(n)) \cong \pi_k(O(n))$ using the long exact sequence of homotopy groups of the fibration $O(n) \hookrightarrow EO(n) \to BO(n)$). The generators of the first eight homotopy groups of O are given by Hopf bundles.

Similarly one can consider stable Stiefel manifolds and stable grassmanians. Let $V_k(\mathbf{R}^\infty) = \lim_{n\to\infty} V_k(\mathbf{R}^n)$ and $G_k(\mathbf{R}^\infty) = \lim_{n\to\infty} G_k(\mathbf{R}^n)$. Then $\pi_i(V_k(\mathbf{R}^\infty)) = \lim_{n\to\infty} \pi_i(V_k(\mathbf{R}^n))$ and $\pi_i(G_k(\mathbf{R}^\infty)) = \lim_{n\to\infty} \pi_i(G_k(\mathbf{R}^n))$. In particular $\pi_i(V_k(\mathbf{R}^\infty)) = 0$.

A project for Chapter 4 was to show that for every topological group G, there is a principal G-bundle $EG \to BG$ where EG is contractible.

This bundle classifies principal G-bundles in the sense that given a principal G-bundle $p : G \hookrightarrow E \to B$ over a CW-complex B (or more generally a paracompact space), there is a map of principal G-bundles

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & EG \arrow[d] \\ B \arrow[r, "f"'] & BG \end{tikzcd}$$

and that the homotopy class $[f] \in [B, BG]$ is uniquely determined. It follows that the (weak) homotopy type of BG is uniquely determined.

Corollary 6.50. *The infinite grassmannian $G_k(\mathbf{R}^\infty)$ is a model for $BO(k)$. The principal $O(k)$ bundle*

$$O(k) \hookrightarrow V_k(\mathbf{R}^\infty) \to G_k(\mathbf{R}^\infty)$$

is universal and classifies principal $O(k)$-bundles. The canonical bundle

$$\mathbf{R}^k \hookrightarrow \gamma_k(\mathbf{R}^\infty) \to G_k(\mathbf{R}^\infty)$$

classifies $\mathbf{R}^k$-vector bundles with structure group $O(k)$ (i.e. $\mathbf{R}^k$-vector bundles equipped with metric on each fiber which varies continuously from fiber to fiber). $\square$

The fact that the grassmannian classifies orthogonal vector bundles makes sense from a geometric point of view. If $M \subset \mathbf{R}^n$ is a k-dimensional smooth submanifold, then for any point $p \in M$, the tangent space T_pM

defines a k-plane in $\mathbf{R}^n$, and hence a point in $G_k(\mathbf{R}^n)$. Likewise a tangent vector determines a point in the canonical bundle $\gamma_k(\mathbf{R}^n)$. Thus there is a bundle map

$$\begin{tikzcd} TM \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & \gamma_k(\mathbf{R}^n) \arrow[d] \\ M \arrow[r, "f"'] & G_k(\mathbf{R}^n) \end{tikzcd}$$

Moreover, $G_k(\mathbf{R}^\infty)$ is also a model for $BGL_k(\mathbf{R})$ and hence is a classifying space for k-plane bundles over CW-complexes. This follows either by redoing the above discussion, replacing k-frames by sets of k-linearly independent vectors, or by using the fact that $O(k) \hookrightarrow GL_k(\mathbf{R})$ is a homotopy equivalence, with the homotopy inverse map being given by the Gram-Schmidt process.

Similar statements apply in the complex setting to unitary groups $U(n)$. Let

$$G_k(\mathbf{C}^n) = \text{complex } k\text{-planes in } \mathbf{C}^n$$

$$G_k(\mathbf{C}^n) = U(n)/(U(k) \times U(n-k)), \quad \text{the complex grassmanian}$$

$$V_k(\mathbf{C}^n) = U(n)/U(n-k), \quad \text{the unitary Stiefel manifold.}$$

There are principal fiber bundles

$$U(n-k) \hookrightarrow U(n) \to V_k(\mathbf{C}^n)$$

and

$$U(k) \hookrightarrow V_k(\mathbf{C}^n) \to G_k(\mathbf{C}^n).$$

Moreover, $V_1(\mathbf{C}^n) \cong S^{2n-1}$, therefore

$$\pi_k(U(n)) \cong \pi_k(U(n-1)) \ \text{ if } \ k < 2n-2$$

and so letting

$$U = \lim_{n\to\infty} U(n),$$

we conclude that

$$\pi_k U = \pi_k(U(n)) \ \text{ for } n > 1 + \frac{k}{2}.$$

Bott periodicity holds for the unitary group; the precise statement is the following.

Theorem 6.51 (Bott periodicity)**.**

$$\pi_k U \cong \pi_{k+2} U \text{ for } k \in \mathbf{Z}_+.$$

Moreover,

$$\pi_k U = \begin{cases} \mathbf{Z} & \text{if } k \text{ is odd, and} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Exercise 108. Prove that $\pi_1 U = \mathbf{Z}$ and $\pi_2 U = 0$.

Taking determinants give fibrations $SO(n) \hookrightarrow O(n) \xrightarrow{\det} \{\pm 1\}$ and $SU(n) \hookrightarrow U(n) \xrightarrow{\det} S^1$. In particular, $SO(n)$ is the identity path–component of $O(n)$, so $\pi_k(SO(n)) = \pi_k(O(n))$ for $k \geq 1$. Similarly, since $\pi_k(S^1) = 0$ for $k > 1$, $\pi_1(SU(n)) = 0$ and $\pi_k SU(n) = \pi_k(U(n))$ for $k > 1$.

Exercise 109. Prove that $SO(2) = U(1) = S^1$, $SO(3) \cong \mathbf{R}P^3$, $SU(2) \cong S^3$, and that the map $p : S^3 \times S^3 \to SO(4)$ given by $(a, b) \mapsto (v \mapsto av\bar{b})$ where $a, b \in S^3 \subset \mathbf{H}$ and $v \in \mathbf{H} \cong \mathbf{R}^4$ is a 2-fold covering map.

Exercise 110. Using Exercise 109 and the facts:

1. $\pi_n S^n = \mathbf{Z}$ (Hopf degree Theorem).
2. $\pi_k S^n = 0$ for $k < n$ (Hurewicz theorem).
3. $\pi_k S^n \cong \pi_{k+1} S^{n+1}$ for $k < 2n - 1$ (Freudenthal suspension theorem).
4. There is a covering $\mathbf{Z} \hookrightarrow \mathbf{R} \to S^1$.
5. $\pi_n S^{n-1} = \mathbf{Z}/2$ for $n > 3$ (this theorem is due to V. Rohlin and G. Whitehead; see Corollary 9.27).

Compute as many homotopy groups of S^n's, $O(n)$, Grassmann manifolds, Stiefel manifolds, etc. as you can.

6.15. Relative homotopy groups

Let (X, A) be a pair, with base point $x_0 \in A \subset X$. Let $p = (1, 0, \cdots, 0) \in S^{n-1} \subset D^n$.

Definition 6.52. The *relative homotopy group (set if $n = 1$) of the pair* (X, A) is

$$\pi_n(X, A, x_0) = [D^n, S^{n-1}, p; X, A, x_0],$$

the set of based homotopy classes of base point preserving maps from the pair (D^n, S^{n-1}) to (X, A). This is a functor from pairs of spaces to sets ($n = 1$), groups ($n = 2$), and abelian groups ($n > 2$).

Thus, representatives for $\pi_n(X, A, x_0)$ are maps $f : D^n \to X$ such that $f(S^{n-1}) \subset A$, $f(p) = x_0$ and f is equivalent to g if there exists a homotopy

$F : D^n \times I \to X$ so that for each $t \in I$, $F(-,t)$ is base point preserving and takes S^{n-1} into A, and $F(-,0) = f$, $F(-,1) = g$.

(Technical note: associativity is easier to see if instead one takes

$$\pi_n(X, A, x_0) = [D^n, S^{n-1}, P; X, A, x_0]$$

where P is one-half of a great circle, running from p to $-p$, e.g.

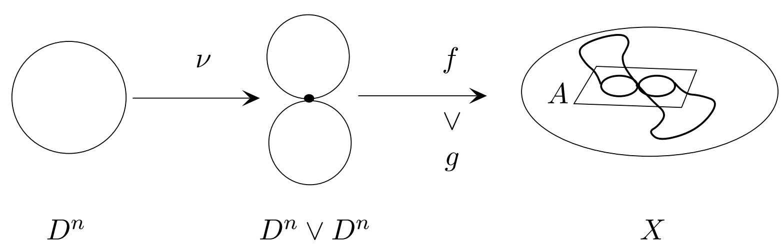
$$P = \{(\cos\theta, \sin\theta, 0, \cdots, 0) \mid \theta \in [0, \pi]\}.$$

This corresponds to the previous definition since the reduced cone on the sphere is the disk.)

Theorem 6.53 (long exact sequence in homotopy of a pair)**.** *The homotopy set $\pi_n(X, A)$ is a group for $n \geq 2$, and is abelian for $n \geq 3$. Moreover, there is a long exact sequence*

$$\cdots \to \pi_n A \to \pi_n X \to \pi_n(X, A) \to \pi_{n-1} A \to \cdots \to \pi_1(X, A) \to \pi_0 A \to \pi_0 X.$$

Proof. The proof that $\pi_n(X, A)$ is a group is a standard exercise, with multiplication based on the idea of the following picture.



Exercise 111. Concoct an argument from this picture and use it to figure out why $\pi_1(X, A)$ is not a group. Also use it to prove that the long exact sequence is exact.

□

Lemma 6.54. *Let $f : E \to B$ be a fibration with fiber F. Let $A \subset B$ be a subspace, and let $G = f^{-1}(A)$, so that $F \hookrightarrow G \xrightarrow{f} A$ is a fibration. Then f induces isomorphims $f_* : \pi_k(E, G) \to \pi_k(B, A)$ for all k. In particular, taking $A = \{b_0\}$ one obtains the commuting ladder*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k F \arrow[r] \arrow[d, "\mathrm{Id}"'] & \pi_k E \arrow[r] \arrow[d, "\mathrm{Id}"'] & \pi_k(E, F) \arrow[r] \arrow[d, "f_*"'] & \pi_{k-1}(F) \arrow[r] \arrow[d, "\mathrm{Id}"'] & \cdots \\
\cdots \arrow[r] & \pi_k F \arrow[r] & \pi_k E \arrow[r] & \pi_k(B) \arrow[r] & \pi_{k-1}(F) \arrow[r] & \cdots
\end{tikzcd}$$

with all vertical maps isomorphisms, taking the long exact sequence of the pair (E, F) to the long exact sequence in homotopy for the fibration $F \hookrightarrow E \to B$.

Proof. This is a straightforward application of the homotopy lifting property. Suppose that $h_0 : (D^k, S^{k-1}) \to (B, A)$ is a map. Viewed as a map $D^k \to B$ it is nullhomotopic, i.e. homotopic to the constant map $c_{b_0} = h_1 : D^k \to B$. Let H be a homotopy, and let $\tilde{h}_1 : D^k \to G \subset E$ be the constant map at the base point of G. Since $f \circ \tilde{h}_1 = h_1 = H(-, 1)$, the homotopy lifting property implies that there is a lift $\widetilde{H} : D^k \times I \to E$ with $f \circ \widetilde{H}(-, 0) = h_0$. This proves that $f_* : \pi_k(E, G) \to \pi_k(B, A)$ is surjective. A similar argument shows that $f_* : \pi_k(E, G) \to \pi_k(B, A)$ is injective.

The only square in the diagram for which commutativity is not obvious is

$$\begin{tikzcd} \pi_k(E,F) \arrow[r] \arrow[d, "f_*"'] & \pi_{k-1}(F) \arrow[d, "\mathrm{Id}"] \\ \pi_k(B) \arrow[r] & \pi_{k-1}(F) \end{tikzcd} \tag{6.4}$$

We leave this as an exercise. □

Exercise 112. Prove that the diagram (6.4) commutes. You will find the constructions in the proof of Theorem 6.39 useful. Notice that the commutativity of this diagram and the fact that f_* is an isomorphism gives an alternative definition of the connecting homomorphism $\pi_k(B) \to \pi_{k-1}(F)$ in the long exact sequence of the fibration $F \hookrightarrow E \to B$.

An alternative and useful perspective on Theorem 6.53 is obtained by replacing a pair by a fibration as follows.

Turn $A \hookrightarrow X$ into a fibration, with A' replacing A and $L(X, A)$ the fiber. Using the construction of Section 6.6 we see that

$$\begin{aligned} L(X, A) &= \{(a, \alpha) \mid \alpha : I \to X, \alpha(0) = a \in A, \alpha(1) = x_0\} \\ &= \mathrm{Map}((I, 0, 1), (X, A, x_0)). \end{aligned}$$

This shows that if $\Omega X \hookrightarrow PX \xrightarrow{e} X$ denotes the path space fibration, then $L(X, A) = PX|_A = e^{-1}(A)$. Thus Lemma 6.54 shows that e induces an isomorphism $e_* : \pi_k(PX, L(X, A)) \to \pi_k(X, A)$ for all k. Since PX is contractible, using the long exact sequence for the pair $(PX, L(X, A))$ gives an isomorphism $\partial : \pi_k(PX, L(X, A)) \xrightarrow{\cong} \pi_{k-1}(L(X, A))$. Therefore the composite

$$\pi_{k-1}(L(X, A)) \xrightarrow{e_* \circ \partial^{-1}} \pi_k(X, A)$$

is an isomorphism which makes the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{k+1}X \arrow[r] \arrow[d, "\mathrm{Id}"'] & \pi_k(L(X,A)) \arrow[r] \arrow[d] & \pi_k A \arrow[r] \arrow[d, "\mathrm{Id}"'] & \pi_k X \arrow[r] \arrow[d, "\mathrm{Id}"'] & \cdots \\
\cdots \arrow[r] & \pi_{k+1}X \arrow[r] & \pi_{k+1}(X,A) \arrow[r] & \pi_k A \arrow[r] & \pi_k X \arrow[r] & \cdots
\end{tikzcd}$$

commute, where the top sequence is the long exact sequence for the fibration $L(X,A) \hookrightarrow A \to X$ and the bottom sequence is the long exact sequence of the pair (X,A).

Homotopy groups are harder to compute and deal with than homology groups, essentially because excision fails for relative homotopy groups. In Chapter 8 we will discuss stable homotopy and generalized homology theories, in which (properly interpreted) excision does hold. Stabilization is a procedure which looks at a space X only in terms of what homotopy information remains in $S^n X$ as n gets large. The fiber $L(X,A)$ and cofiber X/A are stably homotopy equivalent.

6.16. The action of the fundamental group on homotopy sets

The question which arises naturally when studying based spaces is what is the difference between the based homotopy classes $[X,Y]_0$ and the unbased classes $[X,Y]$? Worrying about base points can be a nuisance. It turns out that for simply connected spaces one need not worry; the based and unbased homotopy sets are the same. In general, the fundamental group acts on the based set as we will now explain.

Let X be in $\mathcal{K}_*$, that is, it is a based space with a non-degenerate base point x_0. Suppose Y is a based space.

Definition 6.55. Let $f_0, f_1 : X \to Y$. Let $u : I \to Y$ be a path and suppose there is a homotopy $F : X \times I \to Y$ from f_0 to f_1 so that $F(x_0,t) = u(t)$. Then we say f_0 *is freely homotopic to* f_1 *along* u, and write

$$f_0 \underset{u}{\simeq} f_1.$$

Notice that if $f_0, f_1 : (X,x_0) \to (Y,y_0)$, then u is a loop. Thus a *free* homotopy of *based maps* gives rise to an element of $\pi_1(Y,y_0)$.

Lemma 6.56.

1. *(Existence) Given a map $f_0 : X \to Y$ and a path u in Y starting at $f_0(x_0)$, then $f_0 \underset{u}{\simeq} f_1$ for some f_1.*
2. *(Uniqueness) Suppose $f_0 \underset{u}{\simeq} f_1$, $f_0 \underset{v}{\simeq} f_2$ and $u \simeq v$ (rel ∂I). Then $f_1 \underset{\mathrm{const}}{\simeq} f_2$.*
3. *(Multiplicativity) $f_0 \underset{u}{\simeq} f_1$, $f_1 \underset{v}{\simeq} f_2 \Longrightarrow f_0 \underset{uv}{\simeq} f_2$*

Proof. 1. There exists a free homotopy $F : X \times I \to Y$ with $F(x_0, t) = u(t)$, $F(-, 0) = f_0$, since (X, x_0) is a cofibration:

$$\begin{tikzcd} X \times \{0\} \cup \{x_0\} \times I \arrow[r, "f_0 \cup u"] \arrow[d] & Y. \\ X \times I \arrow[ur, dotted, "F"'] & \end{tikzcd}$$

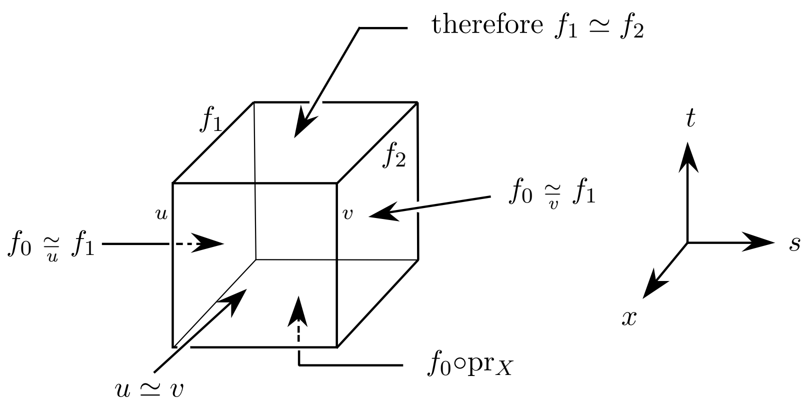
2. Since $(I, \partial I), (X, x_0)$ are cofibrations, so is their product $(X \times I, X \times \partial I \cup x_0 \times I)$ (See Exercise 91) and so the following problem has a solution

$$\begin{tikzcd} X \times I \times \{0\} \cup X \times \{0,1\} \times I \cup \{x_0\} \times I \times I \arrow[r] \arrow[d] & Y. \\ X \times I \times I \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

In this diagram,

1. $X \times I \times \{0\} \to Y$ is the map $(x, s, 0) \mapsto f_0(x)$.
2. $X \times \{0\} \times I \to Y$ is the homotopy of f_0 to f_1 along u.
3. $X \times \{1\} \times I \to Y$ is the homotopy of f_0 to f_2 along v.
4. $\{x_0\} \times I \times I \to Y$ is the path homotopy of u to v.

The situation is represented in the following picture of a cube $X \times I \times I$.



Then $H(-, -, 1)$ is a homotopy of f_1 to f_2 along a constant path.

3. This is clear. □

In light of Lemma 6.56, we can define an action of $\pi_1(Y, y_0)$ on $[X, Y]_0$ by the following recipe.

For $[u] \in \pi_1(Y, y_0)$ and $[f] \in [X, Y]_0$, define $[u][f]$ to be $[f_1]$, where f_1 is any map so that $f \underset{u}{\simeq} f_1$.

Theorem 6.57. *This defines an action of $\pi_1(Y, y_0)$ on the based set $[X, Y]_0$, and $[X, Y]$ is the quotient set of $[X, Y]_0$ by this action if Y is path connected.*

Proof. We need to verify that this action is well-defined. It is independent of the choice of representative of $[u]$ by Lemma 6.56, part 2. Suppose now $[f] = [g] \in [X, Y]_0$ and $g \underset{u}{\simeq} g_1$. Then

$$f_1 \underset{u^{-1}}{\simeq} f \underset{\text{const}}{\simeq} g \underset{u}{\simeq} g_1$$

so that f_1 and g_1 are based homotopic by Lemma 6.56, parts 2 and 3.

This is an action of the group $\pi_1(Y, y_0)$ on the set $[X, Y]_0$ by Lemma 6.56, part 3. Let

$$\Phi : [X, Y]_0 \to [X, Y]$$

be the forgetful functor. Clearly $\Phi([u][f]) = [f]$ and if $\Phi[f_0] = \Phi[f_1]$, then there is a u so that $[u][f_0] = [f_1]$. Finally Φ is onto by Lemma 6.56, part 3 and the fact that Y is path-connected. □

Corollary 6.58. *A based map of path connected spaces is null-homotopic if and only if it is based null-homotopic.*

Proof. If c denotes the constant map, then clearly $c \underset{u}{\simeq} c$ for any $u \in \pi_1 Y$. Thus $\pi_1 Y$ fixes the class in $[X, Y]_0$ containing the constant map. □

Corollary 6.59. *Let $X, Y \in \mathcal{K}_*$. If Y is a path connected and simply-connected space then the forgetful functor $[X, Y]_0 \to [X, Y]$ is bijective.* □

6.16.1. Alternative description in terms of covering spaces. Suppose Y is path connected, and X is *simply connected.* Then covering space theory says that any map $f : (X, x_0) \to (Y, y_0)$ lifts to a *unique* map $\tilde{f} : (X, x_0) \to (\tilde{Y}, \tilde{y}_0)$, where $\tilde{Y}$ denotes the universal cover of Y. Moreover based homotopic maps lift to based homotopic maps. Thus the function

$$p_* : [X, \tilde{Y}]_0 \to [X, Y]_0$$

induced by the cover $p : (\tilde{Y}, \tilde{y}_0) \to (Y, y_0)$ is a *bijection.* On the other hand, since $\tilde{Y}$ is path connected and simply connected, Corollary 6.59 shows that the function $[X, \tilde{Y}]_0 \to [X, \tilde{Y}]$ induced by the inclusion is a bijection.

Now $\pi_1(Y, y_0)$ can be identified with group of covering transformations of $\tilde{Y}$. Thus, $\pi_1(Y, y_0)$ acts on $[X, \tilde{Y}]$ by post composition i.e. $\alpha : \tilde{Y} \to \tilde{Y}$ acts on $f : X \to \tilde{Y}$ by $\alpha \circ f$. (Note: one must be careful with left and right actions: by convention $\pi_1(Y, y_0)$ acts on $\tilde{Y}$ on the right, so $\alpha \circ f$ means the function $x \mapsto f(x) \cdot \alpha$.)

A standard exercise in covering space theory shows that if $\alpha \in \pi_1(Y, y_0)$ the diagram

$$\begin{tikzcd}
{[X,Y]_0} \arrow[d, "\alpha"'] & {[X,\tilde{Y}]_0} \arrow[l, "\cong"'] \arrow[r, "\cong"] & {[X,\tilde{Y}]} \arrow[d, "\alpha"] \\
{[X,Y]_0} & {[X,\tilde{Y}]_0} \arrow[l, "\cong"'] \arrow[r, "\cong"] & {[X,\tilde{Y}]}
\end{tikzcd}$$

commutes, where the action on the left is via an α-homotopy, and the action on the right is the action induced by the covering translation corresponding to α, and the two left horizontal bijections are induced by the covering projection. Thus the two notions of action agree.

Since $\pi_n Y = [S^n, Y]_0$, we have the following corollary.

Corollary 6.60. *For any space Y, $\pi_1(Y, y_0)$ acts on $\pi_n(Y, y_0)$ for all n with quotient $[S^n, Y]$, the set of free homotopy classes.* □

One could restrict to simply connected spaces Y and never worry about the distinction between based and unbased homotopy classes of maps into Y. This is not practical in general, and so instead one can make a dimension-by-dimension definition.

Definition 6.61. We say Y is *n-simple* if $\pi_1 Y$ acts trivially on $\pi_n Y$. We say Y is *simple* if Y is n-simple for all n.

Thus, simply connected spaces are simple.

Proposition 6.62. *If F is n-simple, then the fibration $F \hookrightarrow E \to B$ defines a local coefficient system over B with fiber $\pi_n F$.*

(A good example to think about is the Klein bottle mapping onto the circle.)

Proof. Theorem 6.12 shows that given any fibration, $F \hookrightarrow E \to B$, there is a well-defined homomorphism

$$\pi_1 B \to \left\{ \begin{array}{l} \text{Homotopy classes of self-homotopy} \\ \text{equivalences } F \to F \end{array} \right\}$$

A homotopy equivalence induces a bijection

$$[S^n, F] \xrightarrow{\cong} [S^n, F].$$

But, since we are assuming that F is n-simple, this is the same as an automorphism

$$\pi_n F \to \pi_n F$$

Thus, we obtain a homomorphism

$$\rho : \pi_1 B \to \mathrm{Aut}(\pi_n(F)),$$

i.e. a local coefficient system over B. □

Exercise 113. Prove that the action of $\pi_1(Y, y_0)$ on itself is just given by conjugation, so that Y is 1-simple if and only if $\pi_1 Y$ is abelian.

Exercise 114. Show that a topological group is simple. (In fact H-spaces are simple.)

Theorem 6.63. *The group $\pi_1 A$ acts on $\pi_n(X, A)$, $\pi_n X$, and $\pi_n A$ for all n. Moreover, the long exact sequence of the pair*

$$\cdots \to \pi_n A \to \pi_n X \to \pi_n(X, A) \to \pi_{n-1} A \to \cdots$$

is $\pi_1 A$-equivariant.

Proof. Let $h : (I, 0, 1) \to (A, x_0, x_0)$ represent $u \in \pi_1(A, x_0)$. Let $f : (D^n, S^{n-1}, p) \to (X, A, x_0)$. Then since (S^{n-1}, p) is an NDR–pair, the problem

$$\begin{tikzcd} S^{n-1} \times \{0\} \cup \{p\} \times I \arrow[r, "f_{|S^{n-1}} \cup h"] \arrow[d] & A \\ S^{n-1} \times I \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

has a solution h. Since (D^n, S^{n-1}) is a cofibration, the problem

$$\begin{tikzcd} D^n \times \{0\} \cup S^{n-1} \times I \arrow[r, "f \cup h"] \arrow[d] & X \\ D^n \times I \arrow[ur, dotted, "F"'] & \end{tikzcd}$$

has a solution F. By construction, $F(x, 0) = f(x)$, and also $F(-, 1)$ takes the triple (D^n, S^{n-1}, p) to (X, A, x_0). Taking $u \cdot [f] = [F(-, 1)]$ defines the action of $\pi_1(A, x_0)$ on $\pi_n(X, A; x_0)$. It follows immediately from the definitions that the maps in the long exact sequence are $\pi_1 A$-equivariant. □

Definition 6.64. A *pair* (X, A) is *n-simple* if $\pi_1 A$ acts trivially on $\pi_n(X, A)$ for all n.

6.17. The Hurewicz and Whitehead Theorems

Perhaps the most important result of homotopy theory is the Hurewicz Theorem. We will state the general relative version of the Hurewicz theorem and its consequence, the Whitehead theorem, in this section.

Recall that D^n is oriented as a submanifold of $\mathbf{R}^n$, i.e., the chart $D^n \hookrightarrow \mathbf{R}^n$ determines the local orientation at any $x \in D^n$ via the excision isomorphism $H_n(D^n, D^n - \{x\}) \cong H_n(\mathbf{R}^n, \mathbf{R}^n - \{x\})$. This determines the fundamental class $[D^n, S^{n-1}] \in H_n(D^n, S^{n-1})$. The sphere S^{n-1} is oriented as the boundary of D^n, i.e. the fundamental class $[S^{n-1}] \in H_{n-1}(S^{n-1})$ is defined by $[S^{n-1}] = \delta([D^n, S^{n-1}])$ where $\delta : H_n(D^n, S^{n-1}) \xrightarrow{\cong} H_{n-1}(S^{n-1})$ is the connecting homomorphism in the long exact sequence for the pair (D^n, S^{n-1}).

Definition 6.65. The *Hurewicz map* $\rho : \pi_n X \to H_n X$ is defined by

$$\rho([f]) = f_*([S^n]),$$

where $f : S^n \to X$ represents an element of $\pi_n X, [S^n] \in H_n S^n \cong \mathbf{Z}$ is the generator (given by the natural orientation of S^n) and $f_* : H_n S^n \to H_n X$ the induced map.

There is also a *relative Hurewicz map* $\rho : \pi_n(X, A) \to H_n(X, A)$ defined by

$$\rho([f]) = f_*([D^n, S^{n-1}]).$$

Here $[D^n, S^{n-1}] \in H_n(D^n, S^{n-1}) \cong \mathbf{Z}$ is the generator given by the natural orientation, and $f_* : H_n(D^n, S^{n-1}) \to H_n(X, A)$ is the homomorphism induced by $f : (D^n, S^{n-1}, *) \to (X, A, x_0) \in \pi_n(X, A; x_0)$.

Since the connecting homomorphism $H_n(D^n, S^{n-1}) \xrightarrow{\partial} H_{n-1}(S^{n-1})$ takes $[D^n, S^{n-1}]$ to $[S^{n-1}]$, the map of exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(A) \arrow[r] \arrow[d, "\rho"] & \pi_n(X) \arrow[r] \arrow[d, "\rho"] & \pi_n(X,A) \arrow[r] \arrow[d, "\rho"] & \pi_{n-1}(A) \arrow[r] \arrow[d, "\rho"] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

Let $\pi_n^+(X, A)$ be the quotient of $\pi_n(X, A)$ by the normal subgroup generated by

$$\{x(\alpha(x))^{-1} | x \in \pi_n(X, A),\ \alpha \in \pi_1 A\}.$$

(Thus $\pi_n^+(X, A) = \pi_n(X, A)$ if $\pi_1 A = \{1\}$, or if (X, A) is n-simple.)

Clearly ρ factors through $\pi_n^+(X, A)$, since $f_*([D^n, S^n])$ depends only on the free homotopy class of f. The following theorem is the subject of one of the projects for this chapter. It says that for simply connected spaces, the first non-vanishing homotopy and homology groups coincide. The Hurewicz theorem is the most important result in algebraic topology. We will give a proof the Hurewicz theorem for simply connected spaces in Chapter 10.

Theorem 6.66 (Hurewicz theorem)**.**

1. *Let $n > 0$. Suppose that X is path-connected. If $\pi_k(X, x_0) = 0$ for all $k < n$, then $H_k(X) = 0$ for all $0 < k < n$, and the Hurewicz map*
$$\rho : \pi_n X \to H_n X$$
is an isomorphism if $n > 1$, and a surjection with kernel the commutator subgroup of $\pi_1 X$ if $n = 1$.
2. *Let $n > 1$. Suppose X and A are path-connected. If $\pi_k(X, A) = 0$ for all $k < n$ then $H_k(X, A) = 0$ for all $k < n$, and*
$$\rho : \pi_n^+(X, A) \to H_n(X, A)$$
is an isomorphism. In particular $\rho : \pi_n(X, A) \to H_n(X, A)$ is an epimorphism.

□

Corollary 6.67 (Hopf degree theorem)**.** *The Hurewicz map $\rho : \pi_n S^n \to H_n S^n$ is an isomorphism. Hence a degree zero map $f : S^n \to S^n$ is null-homotopic.* □

Although we have stated this as a corollary of the Hurewicz theorem, it can be proven directly using only the (easy) simplicial approximation theorem. (The Hopf degree theorem was covered as a project in Chapter 5.)

Definition 6.68.

1. A space X is called *n-connected* if $\pi_k X = 0$ for $k \leq n$. (Thus "simply connected" is synonymous with 1-connected).
2. A pair (X, A) is called *n-connected* if $\pi_k(X, A) = 0$ for $k \leq n$.
3. A map $f : X \to Y$ is called *n-connected* if the pair (M_f, X) is n-connected, where M_f = mapping cylinder of f.

Using the long exact sequence for (M_f, X) and the homotopy equivalence $M_f \sim Y$ we see that f is *n-connected* if and only if
$$f_* : \pi_k X \to \pi_k Y$$
is an isomorphism for $k < n$ and an epimorphism for $k = n$. Replacing the map $f : X \to Y$ by a fibration and using the long exact sequence for the

homotopy groups of a fibration shows that f is n-connected if and only if the homotopy fiber of f is $(n-1)$-connected.

Corollary 6.69 (Whitehead theorem)**.**

1. *If $f : X \to Y$ is n-connected, then $f_* : H_q X \to H_q Y$ is an isomorphism for all $q < n$ and an epimorphism for $q = n$.*
2. *If X, Y are 1-connected, and $f : X \to Y$ is a map such that*
$$f_* : H_q X \to H_q Y$$
is an isomorphism for all $q < n$ and an epimorphism for $q = n$. Then f is n-connected.
3. *If X, Y are 1-connected spaces, $f : X \to Y$ a map inducing an isomorphism on $\mathbf{Z}$-homology, then f induces isomorphisms $f_* : \pi_k X \xrightarrow{\cong} \pi_k Y$ for all k.*

Exercise 115. Prove Corollary 6.69.

A map $f : X \to Y$ inducing an isomorphism of $\pi_k X \to \pi_k Y$ for all k is called a *weak homotopy equivalence*. Thus a map inducing a homology isomorphism between simply connected spaces is a weak homotopy equivalence. Conversely a weak homotopy equivalence between two spaces gives a homology isomorphism.

We will see later (Theorem 7.34) that if X, Y are CW-complexes, then $f : X \to Y$ is a weak homotopy equivalence if and only if f is a homotopy equivalence. As a consequence,

Corollary 6.70. *A continuous map $f : X \to Y$ between simply connected CW-complexes inducing an isomorphism on all $\mathbf{Z}$-homology groups is a homotopy equivalence.* □

This corollary does not imply that if X, Y are two simply connected spaces with the same homology, then they are homotopy equivalent; one needs a *map* inducing the homology equivalence.

For example, $X = S^4 \vee (S^2 \times S^2)$ and $Y = \mathbf{C}P^2 \vee \mathbf{C}P^2$ are simply connected spaces with the same homology. They are not homotopy equivalent because their cohomology rings are different. In particular, there does not exist a continuous map from X to Y inducing isomorphisms on homology.

The Whitehead theorem for non-simply connected spaces involves homology with local coefficients: If $f : X \to Y$ is a map, let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the corresponding lift to universal covers. Recall from Shapiro's lemma (Exercise 75) that

$$H_k(\tilde{X}; \mathbf{Z}) \cong H_k(X, \mathbf{Z}[\pi_1 X]) \quad \text{for all } k$$

and

$$\pi_k\tilde{X} \cong \pi_k X \quad \text{for } k > 1$$

(and similarly for Y).

We obtain (with $\pi = \pi_1 X \cong \pi_1 Y$):

Theorem 6.71. *If $f : X \to Y$ induces an isomorphism $f_* : \pi_1 X \to \pi_1 Y$, then f is n-connected if and only if it induces isomorphisms*

$$H_k(X; \mathbf{Z}[\pi]) \to H_k(Y; \mathbf{Z}[\pi])$$

for $k < n$ and an epimorphism

$$H_n(X; \mathbf{Z}[\pi]) \to H_n(Y; \mathbf{Z}[\pi])$$

In particular, f is a weak homotopy equivalence (homotopy equivalence if X, Y are CW-complexes) if only if $f_* : H_k(X; A_\rho) \to H_k(Y; A_\rho)$ is an isomorphism for all local coefficient systems $\rho : \pi \to \mathrm{Aut}(A)$. □

Thus, in the presence of a map $f : X \to Y$, *homotopy equivalences can be detected by homology.*

6.18. Projects for Chapter 6

6.18.1. The Hurewicz theorem. The statement is given in Theorem 6.66. A reference is §IV.4-IV.7 in [**43**]. Another possibility is to give a spectral sequence proof. Chapter 10 contains a spectral sequence proof the Hurewicz theorem.

6.18.2. The Freudenthal suspension theorem. The statement is given in Theorem 8.7. A good reference for the proof is §VII.6-VII.7 in [**43**]. You can find a spectral sequence proof in Section 10.3.