PROBLEME C

1) Simplifier les sommes suivantes

i) $\sum_{k=2}^{n} 3^{-2k}$

ii) $\sum_{k=0}^{n^2-1} \frac{1}{\sqrt{k+1}+\sqrt{k}}$

iii) $\sum_{k=0}^{n} \frac{2^k}{(1+2^k)(1+2^{k+1})}$

2) Montrer que $\forall n \in \mathbb{N}$, on a

i) $\sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

ii) $\sum_{k=0}^{n} (-1)^k k = \frac{(-1)^n(2n+1)-1}{4}$

PROBLEME D

BONUS

1) Montrer que $\binom{n}{k}\binom{n-k}{p-k} = \binom{p}{k}\binom{n}{p}$ $p \leq n$

2) En deduire que $\sum_{k=0}^{p} \binom{n}{k}\binom{n-k}{p-k} = 2^p\binom{n}{p}$

3) " " " $\sum_{k=0}^{p} (-1)^k \binom{n}{k}\binom{n-k}{p-k} = 0$

FIN BONNE CHANCE

☺

①

PROBLEME C

i) $\sum_{k=2}^{n} 3^{-2k} = \sum_{k=2}^{n} (3^{-2})^k = \sum_{k=2}^{n} \left(\frac{1}{9}\right)^k = \frac{1}{9^2} \frac{1-\left(\frac{1}{9}\right)^{n-2+1}}{1-\frac{1}{9}} = \dots$

ii) $\sum_{k=0}^{n^2-1} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sum_{k=0}^{n^2-1} \frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k+1})^2-(\sqrt{k})^2}$

$= \sum_{k=0}^{n^2-1} (\sqrt{k+1}-\sqrt{k}) = \sqrt{n^2} - \sqrt{0} = n$

iii) $\sum_{k=0}^{n} \frac{2^k}{(1+2^k)(1+2^{k+1})} = \sum_{k=0}^{n} \frac{2^{k+1}-2^k}{(1+2^k)(1+2^{k+1})}$

$= \sum_{k=0}^{n} \frac{(2^{k+1}+1)-(2^k+1)}{(1+2^k)(1+2^{k+1})} = \sum_{k=0}^{n} \frac{\cancel{2^{k+1}+1}}{(1+2^k)\cancel{(1+2^{k+1})}} - \frac{\cancel{2^k+1}}{\cancel{(1+2^k)}(1+2^{k+1})}$

$= \sum_{k=0}^{n} \left(\frac{1}{1+2^k} - \frac{1}{1+2^{k+1}}\right) = \frac{1}{1+2^0} - \frac{1}{1+2^{n+1}} = \dots$

② i) par recurrence, posons $S_n = \sum_{k=0}^{n} k^3$

et Montrons que $S_n = \frac{n^2(n+1)^2}{4}$

pour $n=0$, $S_0 = 0 = \frac{0^2 \times 1^2}{4}$

2) i) Supposons $S_n = \dfrac{n^2(n+1)^2}{4}$

donc $S_{n+1} = S_n + (n+1)^3$

$$= \frac{n^2 (n+1)^2}{4} + (n+1)^3$$

$$= (n+1)^2 \left(\frac{n^2}{4} + n+1\right) = (n+1)^2 \left(\frac{n^2+4n+4}{2}\right)$$

$$= \frac{(n+1)^2 (n+2)^2}{4} \quad \text{CQFD}$$

PAS ENCORE

ii) Posons $S_n = \sum_{k=0}^{n} (-1)^k k$ et Mq $S_n = \dfrac{(-1)^n(2n+1)-1}{4}$

pour $n=0$ vrai

supposons $S_n = \dfrac{(-1)^n (2n+1)-1}{4}$

donc $S_{n+1} = S_n + (-1)^{n+1}(n+1)$

$$= \frac{(-1)^n (2n+1)-1}{4} + (-1)^{n+1} (n+1)$$

$$= \frac{(-1)^n (2n+1) - 1 + 4 \cdot (-1)^{n+1} (n+1)}{4}$$

$$= \frac{(-1)^n \left(2n+1 - 4(n+1)\right) - 1}{4}$$

$$= \frac{(-1)^n (-2n-3)-1}{4} = \frac{(-1)^{n+1}(2n+3)-1}{4}$$

CQFD

PROBLEME D

1) $$\binom{n}{k}\binom{n-k}{p-k} = \frac{n!}{k!\,\cancel{(n-k)!}}\,\frac{\cancel{(n-k)!}}{(p-k)!\,\big((n-k)-(p-k)\big)!} = \frac{n!}{k!(p-k)!(n-p)!}$$

$$=$$

$$\binom{p}{k}\binom{n}{p} = \frac{\cancel{p!}}{k!(p-k)!}\,\frac{n!}{\cancel{p!}(n-p)!} = \frac{n!}{k!(p-k)!(n-p)!}$$

2) $$\sum_{k=0}^{p}\binom{n}{k}\binom{n-k}{p-k} = \sum_{k=0}^{p}\binom{p}{k}\binom{n}{p}$$

$$= \binom{n}{p}\sum_{k=0}^{p}\binom{p}{k} = \binom{n}{p}(1+1)^p = \binom{n}{p}2^p$$

3) $$\sum_{k=0}^{p}(-1)^k\binom{n}{k}\binom{n-k}{p-k} = \sum_{k=0}^{p}(-1)^k\binom{p}{k}\boxed{\binom{n}{p}}$$

$$= \binom{n}{p}\sum_{k=0}^{p}(-1)^k\binom{p}{k}$$

$$= \binom{n}{p}(1-1)^p = 0$$